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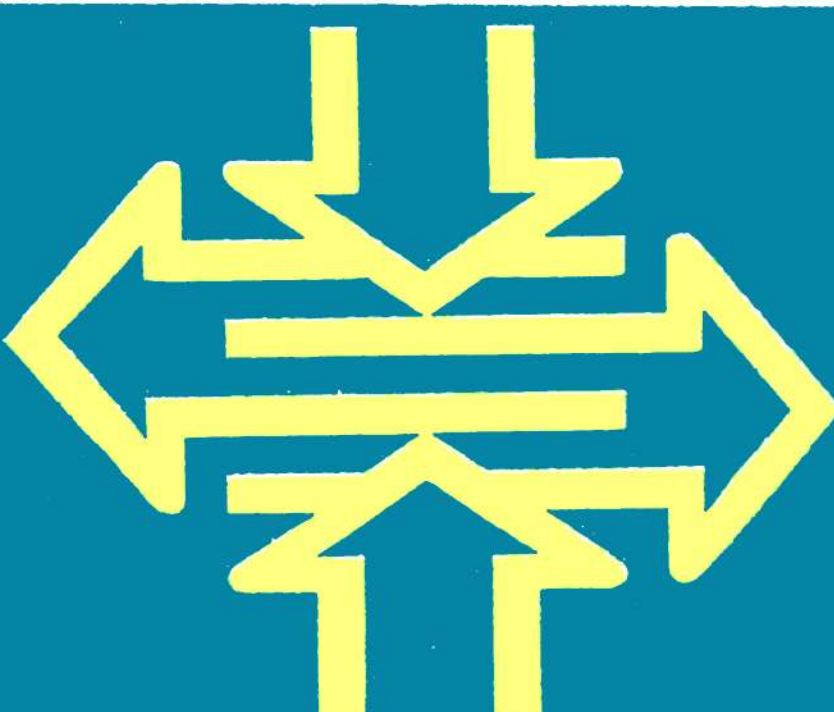

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Strength of Materials

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Основы сопротивления материалов для чертежников-конструкторов

Москва «Машиностроение»

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Strength of Materials

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Contents

Chapter One

Basic Concepts

1.1. The Science of Strength of Materials and Brief Notes on Its Historical Development

All structures (such as buildings, bridges, machines, instruments, etc.) are designed so as to satisfy the requirements of strength, rigidity and stability, which is essential for their reliable and safe operation. All structural elements can deform under the action of external forces, i.e. their shape and dimensions can change. In this process, internal elastic forces appear in the elements, which tend to resist the deformation and return particles of a body into their initial positions. The appearance of elastic forces in a body is due to the existence of internal forces of molecular interaction.

The deformation of a body can disappear partially or fully when the external forces causing this deformation are removed. Deformations which disappear upon removal of a force are called elastic and the property of bodies to restore their initial shape on unloading is called elasticity.

Deformations which remain in a body upon unloading are called residual, or plastic, and the property of bodies to retain residual deformations is called plasticity.

Strength is understood as the abilty of a structure or its elements to withstand a specified load without failure.

Strength calculations are aimed at establishing the minimal required dimensions of structural elements which prevent any possibility of failure under the action of specified loads.

Rigidity (or stiffness) is understood as the capability of a body or structure to resist deformation. In rigidity calculations, the dimensions of a structural element are determined so that the changes of the shape and size of the element under the action of working loads were within

certain limits and could not disturb normal operation of a structure.

Stability is meant as the capability of a structure to resist the forces which tend to move it from the initial state of equilibrium. Stability calculations should ensure that the elements of a structure will retain the initial (design) form of equilibrium under the action of working loads.

The science that studies the principles and methods of calculation of structural elements for strength, rigidity and stability is called strength of materials. The analytical and experimental methods of strength of materials enable us to choose the kind of material which can be employed rationally in particular structural elements and to select properly the shape and dimensions of cross sections of structural elements to ensure their reliable operation with the least expenditure of the material. The methods of strength of materials are also often resorted to for checking whether the dimensions of a designed structure have been chosen correctly or whether the loads acting on particular structural elements are safe (allowable).

Solutions of problems in strength of materials are based on wide use of mathematics. Mathematical methods, however, cannot always describe properly the phenomena and processes that take place in materials and structural elements under the action of applied loads, especially when the bodies being calculated have an intricate shape. Problems of strength of materials for such bodies are solved by experimental methods: polarization-optical (photo-elastic), radiographic, holographic, extensometric, etc. In some cases, models of a structure or its elements are made and tested in order to obtain data on the pattern and magnitude of deformations, since analytical methods turn out to be inapplicable in these cases.

The course on strength of materials is closely associated with the course on theoretical mechanics. The difference between these courses consists in that theoretical mechanics is based on the general laws of mechanical motion and equilibrium of bodies which are considered to be absolutely rigid, i.e. undeformable, whereas

strength of materials takes into account the real properties of materials in structures, in particular, their deformability under the action of external loads.

The methods used in strength of materials are developed and improved continuously.

The first theoretical and experimental studies on the strength of structures which have come to our time were undertaken by Leonardo da Vinci, an Italian scientist, engineer and artist (1452-1519). His works remained unknown for a long time and were interpreted only at the end of the 19th century.

Galileo Galilei, another Italian scientist (1564-1642) is recognized as the founder of the modern theory of strength of materials. In one of his works, dated to 1638, he solved the problem of the bending strength of beams, depending on their dimensions and loads. The statement of the problem and the strength tests of beams, which were carried out by Galileo, gave a strong impulse for development of the science of strength of structures.

M. V. Lomonosov, a prominent Russian scientist (1711-1765), in his treatises on hardness, correlated this concept to the concept of internal forces. He studied experimentally the strength of compressed stands. The theoretical calculation of such stands was made by Leonard Euler, another Russian scientist (1707-1783).

Large contributions to the science of strength were done by French engineers and mathematicians C.L.Navier, A. L. Cauchy, S. D. Poisson, and J. A. Bresse. In 1826, Navier (1785-1836) published a book where the theory of strength of materials was disclosed for the first time and the author made a number of important theoretical conclusions.

It is worth to mention a number of Russian scientists of the 19th century who dealt with problems of strength. Academician M. V. Ostrogradsky (1801-1861) solved some problems in the theory of elasticity. Academician A. V. Gadolin (1828-1892) made extensive studies of the strength of gun barrels. D. I. Zhuravsky, an engineer and bridge constructor (1821-1891) developed a scientifically based laboratory method for determining the properties of various materials and carried out original

studies in the theory of bending of beams and bridge trusses. At the end of the last century, important works in the stability of bar structures were done by F.S. Yasinsky (1856-1899).

Complex problems associated with the strength and stability of ship structures were analysed in the works of I. G. Bubnov (1872-1919). Academician A. N. Krylov (1863-1945), known by his works in ship-building, arrived at exceptionally important solutions in the field of engineering calculations of vibrations caused by variable loads. The works of B. G. Galerkin (1871-1945) relate mainly to the analysis of plates and shells. The method of solution of differential equations proposed by him is widely used in the applied theory of elasticity. Some problems of the theory of impact and stability were enlightened in the works of A. N. Dinnik (1876-1950).

Important scientific investigations in strength of materials were done by Soviet scientists A. A. Ilyushin, Yu.N. Rabotnov, S.V. Serensen, E.I. Grigolyuk, V.V. Bolotin, S.D. Ponomarev, A.R. Rzhanitsyn, et al. An appreciable contribution to theoretical and experimental studies in the strength of structures was done by V. I. Feodosyev whose textbook on strength of materials served as a reference book to several generations of Soviet engineers.

1.2. Kinds of Structural Elements and Loads

All structures consist of structural elements which can be reduced to a relatively small number of principal forms: bars, plates, shells, and solid units.

A bar is a body whose size along one of the axes (length) exceeds many times other dimensions (Fig. 1). The geometrical locus of the centres of gravity of all sections of a bar is the bar axis (line *ABCD* in Fig. 1).

Depending on the shape of axis and the cross sections, bars may be straight, curved, with a constant, continuously varying or stepwise varying cross section (Fig. 2). Examples of straight bars are beams and columns and of curved ones, arches and hooks.

Plates and shells can be characterized by a small thickness compared with other dimensions (Fig. 3). Foundation plates and flat bottoms of vessels are examples of plates. Boilers, tanks, reservoirs and aircraft fueselages are examples of shells.

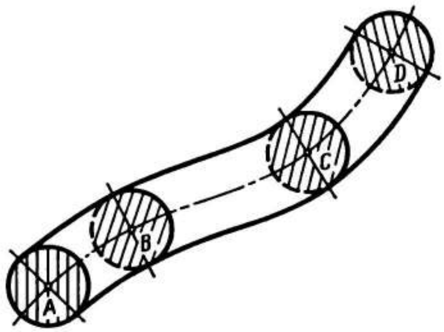

Fig. 1.
Scheme of a bar

A body is called massive if all its dimensions are roughly of the same order (Fig. 4). Examples are foundations of structures, retaining walls, bridge supports (piers), etc. The principal structural element in strength

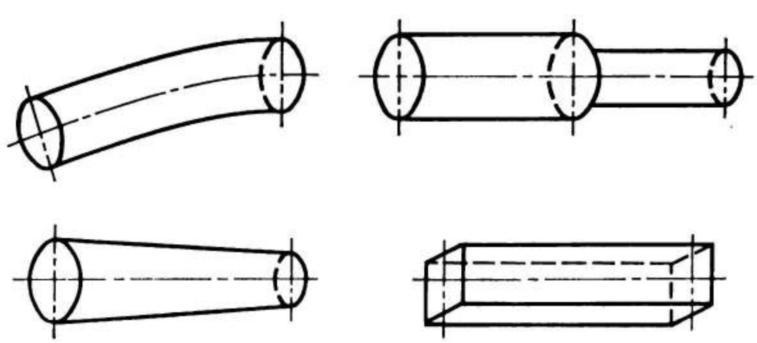

Fig. 2.
Shapes of bars

of materials is a bar which may be called a rod, beam or shaft, depending on the purpose and design.

External forces acting on structural elements are called loads. The external forces which act on a structural

element considered include all force actions from the adjacent structural elements of the structure or from any bodies interacting with that structure. Examples of force actions on a structural element are the pressure of

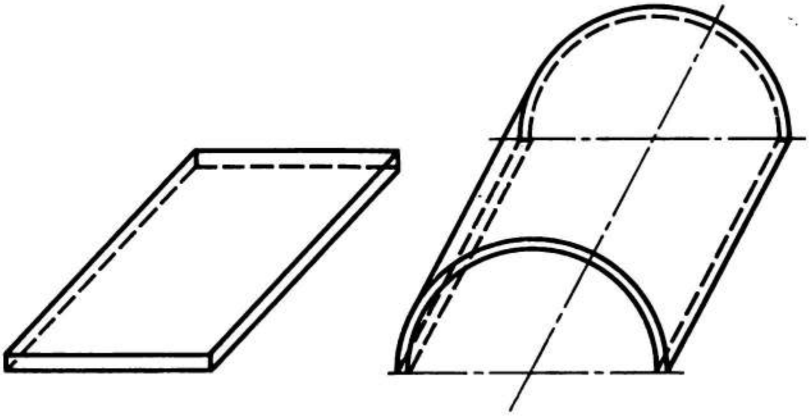

Fig.3.
Schemes of a plate and shell

steam on a piston, the pressure of a liquid on the bottom of a tank. The external forces acting on structural elements also include the weight of the element proper and the reactions of supports (if any).

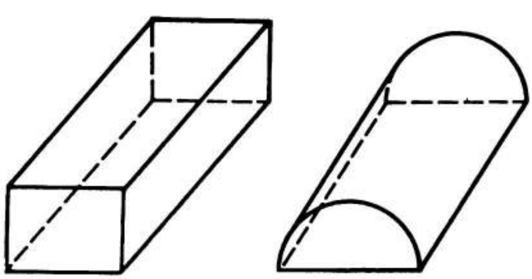

Fig. 4.
Schemes of massive bodies

External forces can be classified by a number of features. Forces which are applied to a body on contact of this body with other bodies are called surface forces (the pressure of wheels on rails, the pressure of water on a dam, etc.). Forces which are distributed over the

volume of a body and applied to internal points of the body are called volume forces. Examples are the gravity forces of a structure or its portion, and inertia forces appearing in accelerated motion of a body.

By the method of application, it is distinguished between concentrated and distributed forces. The former are considered to be applied at a point and are vector quantities. Since a force appears due to interaction of bodies and the pressure is transferred from one body to another through the surface of contact which has certain dimensions, concentrated forces are actually non-existent. In cases when the surface of contact is small compared with the dimensions of a body (structural element), the force can be considered to be applied at a point, in particular in the centre of the contact surface. This assumption can substantially simplify calculations. Concentrated forces are measured in newtons. Distributed forces are those which act over a certain relatively large surface of a structure. These loads are measured in pascals (1 Pa = 1 N/m^2). If forces are distributed along the length of a structural element, they are related per unit length. Examples of distributed forces (or loads) are the pressure of water on a dam or the pressure of a gas on the walls of a vessel.

By the pattern of their variation in time, loads can be divided into static and dynamic. By a static action is meant such an action when the load is applied gradually to a structural element and increases from zero to the maximum value and then remains constant or almost constant during a more or less long interval of time. Examples of static loads are centrifugal forces acting on a rotor during acceleration and subsequent uniform rotation. Dynamic loads are characterized by quick variations in time of their magnitude, direction or point of application. An example of such a load is the impact of a hammer in pile driving.

The forces acting on a structure or its elements are mostly determined by calculations, for instance, the pressure of gases in a vessel, gravity forces, or inertia forces.

In some cases, active forces cannot be calculated quite accurately for some or other reasons, say, because

of vibration processes involved, and the sole method for determining them is direct measurement.

1.3. The Concept of Design Diagram. Principal Hypotheses and Assumptions in Strength of Materials

The design diagram is a simplified or conditional graphical representation of a structure and the forces acting on it. In the selection of a design diagram, all secondary features and factors which have no effect on the strength of a structure are disregarded.

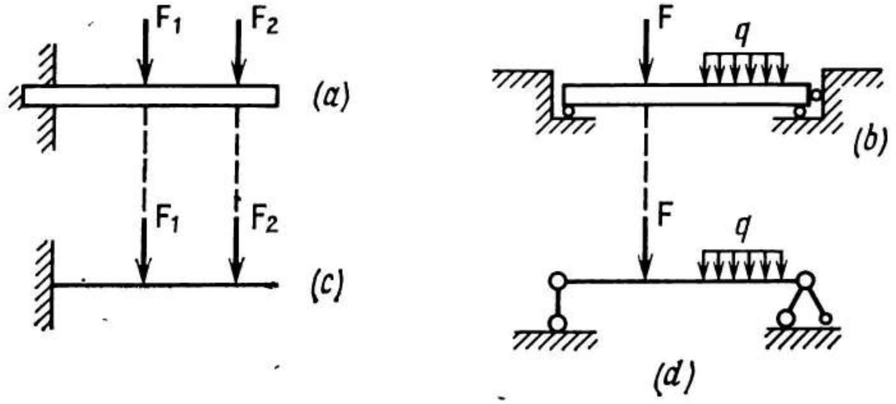

Fig.5.
Loading diagrams and design lagrams

The concept of design diagram can simplify substantially the general calculation methods. For instance, Fig. 5, *a* and *b*, shows two beams with their loads, and Fig. 5, *c* and *d*, the corresponding design diagrams. The design diagram of each beam is given as an axial line with idealized supports. In construction of design diagrams, some deviations from the real conditions of the operation of structures are permissible. For instance, it may be assumed that the load acting on a beam is transmitted to the axis, because of which the point of load application in a design diagram can be determined as shown in Fig. 5, *c* and *d*.

Several design diagrams may be constructed for one and the same structure, depending on what object is to be analysed. On the other hand, several similar structures may have a single design diagram. This makes it possible to obtain solutions for a whole class of engineering problems by studying a single design diagram.

Problems in strength of materials are mostly solved by resorting to certain assumptions and hypotheses on the properties of materials and nature of deformation. These assumptions are such that the conclusions made on their basis agree quite closely with the results of experimental tests. The principal assumptions are as follows.

1. The material of a structure is considered to be homogeneous in structure and continuous at all points of the body. A homogeneous structure implies that any however small particles of a body possess the same properties. Among the materials that are considered homogeneous are metals and alloys, such as steels, aluminium, copper, etc.

2. All bodies are assumed to be absolutely elastic, i.e. their deformations disappear completely upon removal of the load. Actually, this is true only up to a definite value of load.

3. Strength of materials considers all material to be isotropic, i.e. possessing the same properties in all directions. Isotropic materials include metals, concrete, and some plastics. Materials possessing different properties in various directions are called anisotropic. Examples are wood, reinforced plastics, etc.

4. Deformations of elastic bodies under the action of external loads are small compared with the dimensions of bodies, i.e. the dimensions of a body are not changed substantially on elastic deformation. This assumption simplifies substantially the calculations, since it makes it possible to neglect changes in the arrangement of the active forces on deformation.

5. Since deformations considered in strength of materials are small, it can be assumed that external forces act independently from one another, i.e. the deformations and internal forces appearing in elastic bodies do not depend on the order in which the external forces are applied. Besides, it is assumed that the total effect of

the whole system of forces acting on a body is the sum of the effects produced by each force separately.

This assumption is known as the principle of superposition. For instance, for the beam shown in Fig. 6, the

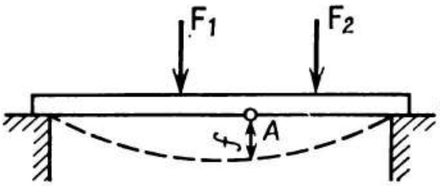

Fig. 6.
A beam loaded by two concentrated forces

deflection at point A under the action of forces F_1 and F_2 is determined as

$$f = f_1 + f_2$$

where f_1 and f_2 are the deflections of the beam at point A due to each force separately.

6. Cross sections of a bar, if they have been plane and normal to the axis before deformation, remain such after deformation.

Chapter Two

Principles of Statics

2.1. Problems and Axioms of Statics. Constraints and Their Reactions

Statics is the part of mechanics that is devoted to the general study of forces acting on material bodies and the conditions of equilibrium of these bodies under the action of forces.

A combination of forces acting simultaneously on a body is called a system of forces. Those systems which produce the same effect on a body are called equivalent.

An equilibrium state of a body is the state in which the body remains at rest relative to other material bodies. Statics considers only what is called the absolute equilibrium of bodies, i.e. the state of equilibrium when the motion of the body in question can be neglected. In practical engineering calculations, the equilibrium of a body relative to the Earth or to other bodies which are rigidly fixed with the Earth may be regarded to be absolute.

Static assumes that all bodies are absolutely rigid, i.e. those in which distances between any two points always remain the same.

The principal problems solved in statics are as follows:

(1) composition of forces and reduction of a system of forces acting on a body to the simplest form;

(2) determination of the conditions of equilibrium for a system of forces applied to a solid body.

These problems can be solved either graphically by means of geometrical constructions or analytically, i.e. by numerical calculations.

All theorems and equations of statics are based on a few propositions which are taken without mathematical proof and called the axioms of statics.

Axiom 1. An absolutely rigid body is in equilibrium under the action of two forces if only these forces are equal in magnitude and directed oppositely along the same line (Fig. 7).

Axiom 2. The effect of a particular system of forces on an absolutely rigid body will not be disturbed if an arbitrary balanced system of forces is added to or subtracted from that system.

A system of forces is called a balanced system if the body to which this system is applied continues to be at rest. This axiom states that if two systems of forces differ from each other by a balanced system, they are equivalent.

A corollary of axioms 1 and 2: the effect of a force on an absolutely rigid body will not be changed if the point of application of the force is transferred into any other point on the line of force application.

Let a force **F** be applied in point A to a body (Fig. 8). Let two forces $\mathbf{F}_1$ and $\mathbf{F}_2$, which are equal in magnitude to

force **F** and directed oppositely along the line of application of that force, be then applied to the body. According to the first axiom of statics, the forces $\mathbf{F}_1$ and $\mathbf{F}_2$ are mutually counter-balanced, and therefore, by virtue of axiom 2, their application will not change the original state of the body. The result is a system of three forces, **F**, $\mathbf{F}_1$,

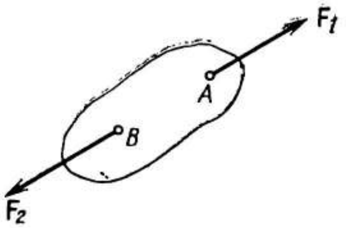

Fig. 7
Equilibrium of a body

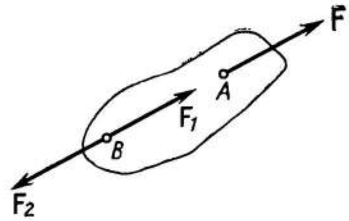

Fig. 8.
Transfer of the point of force application

and $\mathbf{F}_2$, which are applied along the same line. The forces **F** and $\mathbf{F}_2$ can however be cancelled since they counterbalance each other, after which there remains the single force $\mathbf{F}_1$. It is equal in magnitude to the force **F** and directed in the same sense, but is applied in the new point *B* which has been transferred, as it were, along the line of application of the initial force **F**.

This result is true only for forces applied to absolutely rigid bodies and can be used in engineering calculations only when determining the equilibrium conditions of a structure, but not when it is required to analyse the internal forces appearing in the elements of the structure.

For instance, the bar *AB* as shown in Fig. 9*a*, will be equilibrium if the forces $\mathbf{F}_1$ and $\mathbf{F}_2$ applied to it are equal. The equilibrium will not be disturbed if the points of application of both forces are transferred into a new point of the bar (say, point *C* in Fig. 9*b*) or if the point of application of force $\mathbf{F}_1$ is transferred to point *B* and that of force $\mathbf{F}_2$, to point *A* (Fig. 9*c*). The bar will, however, be tensioned by the forces applied in the first case, will not change its dimensions in the second case and will be compressed in the third case. Thus, the point of applica-

tion of a force should not be transferred along the application line if internal forces are to be determined.

Axiom 3. A force which is equivalent to a system of forces is called the resultant of that system. The resultant

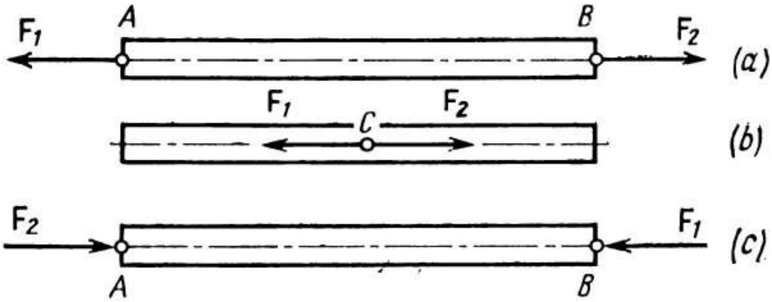

Fig. 9.
Equilibrium of a rod

of two forces applied in a single point at an angle to each other is applied in the same point and represented graphically as the diagonal of the parallelogram constructed on these forces as the sides. The resultant obtained in this way is the vector or geometrical sum of the component forces. Denoting by **R** the resultant of two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ applied in the same point *A* of a body (Fig. 10), it can be written in accordance with the axiom:

$$\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2$$

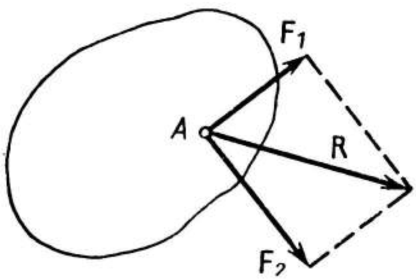

Fig. 10.
Determining the resultant force

where $\mathbf{R}$, $\mathbf{F}_1$ and $\mathbf{F}_2$ are the force vectors. If it is required to find the resultant of two non-parallel forces applied in different points of a body, these forces can first be transferred into the point where their application lines intersect and then the resultant can be found by the rule of parallelogram.

Axiom 4. The equilibrium of a deformable body under the action of a given system of forces will not be disturbed

if the body is considered absolutely rigid. The axiom may be called the rigidity principle. It permits the equilibrium conditions established by the method of statics for absolutely rigid bodies to be applied to any deformable bodies and structures. If the number of equations obtained in this way is insufficient for solving a problem, additional equations can be compiled to take into account the conditions of equilibrium of individual elements of a structure or their deformations. This axiom is used widely in practical engineering calculations of structures.

A body is called free if it can perform all probable movements in the space under the action of the forces applied to it . A body is termed non-free if its movements are restricted by other bodies which are attached to or in contact with that body. The bodies which restrict the freedom of movement of a given body are called its constraints. For instance, a table is a constraint for a body lying on it; a door is constrained by the hinges by which it is attached to the door-post; a rotating shaft is constrained by bearings.

As a body tends under the action of applied forces to perform a movement that is prevented by a constraint, it exerts a certain force on the constraint. By the principle of equality between action and counteraction, the constraint will in turn act on the body. The force that a given constraint exerts on a body and prevents the body's movements is called the force of reaction of the constraint, or simply the constraint reaction. The constraint reaction is equal in magnitude to the force of pressure on the constraint and directed in the opposite sense to the movement that the body would have without the constraint.

All forces except for constraint reactions are called active forces. A characteristic feature of an active force is that its magnitude and direction do not depend directly on other forces acting on a body. On the contrary, constraint reactions differ from active forces in that their magnitudes always depend on these forces and are not known in advance. If active forces have no effect on a constraint, the reaction of this constraint is equal to zero. If a constraint prevents movements of a body

simultaneously in several directions, the direction of the constraint reaction is not known in advance and must be found by calculation.

Problems on equilibrium of non-free bodies are solved in statics by using the following axiom: any non-free body can be regarded as a free body if its constraints are rejected and their effect is replaced by the reactions of these constraints. Thus, with the aid of this axiom, the equilibrium conditions established for a free body can be

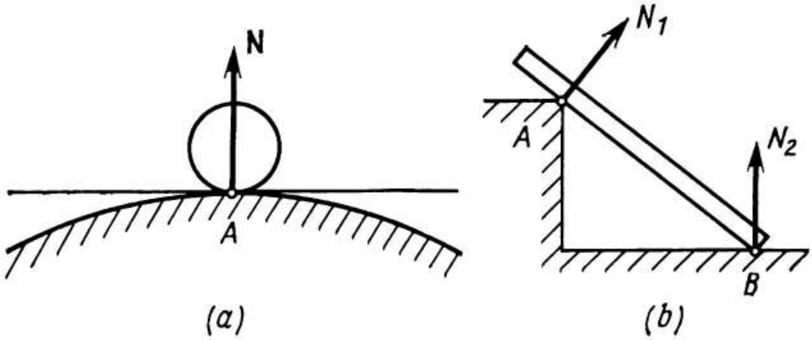

Fig. 11.
Constraint reactions

applied to a non-free body, if the forces acting on the body are supplemented with the reactions forces of the rejected constraints.

Determination of constraint reactions is of large practical importance, since the knowledge of constraint reactions permits us to determine the forces of pressure on the constraints, i.e. the initial data essential for strength calculations of structural members. Let us show how the directions of reactions of the main types of constraints can be determined.

1. A body bears in point A against an immobile surface (Fig. 11*a*). In that case, and in the absence of friction, the reaction of the support surface is applied to the body in point A and directed normally to the support surface in that point. This force is usually denoted by $\mathbf{N}$ and called a normal reaction.

If one of the contacting surfaces is a point, say a bar supported in points A and B (Fig. 11*b*), the reactions $\mathbf{N}_1$

and $\mathbf{N}_2$ are directed along normals to the other surface, as illustrated in the figure.

2. A body (weight) G is constrained by a flexible unstretchable filament which prevents the body to move from point C along the filament line CA and from point B along the filament line BA (Fig. 12). Consequently, the reactions $\mathbf{T}_1$ and $\mathbf{T}_2$ of this constraint are directed along the filament. It is clear that the filament in this example can be replaced by a rope, chain, cable, etc.

3. A cylindrical hinge or a hingedly fixed support. An example of a hinged connection (or simply of a hinge)

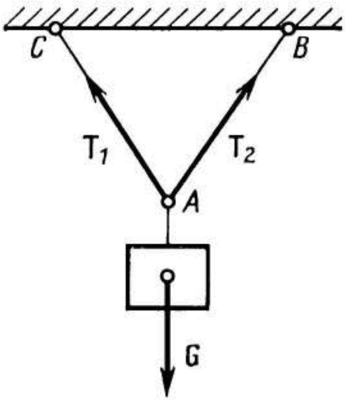

Fig. 12.
Constraint reaction

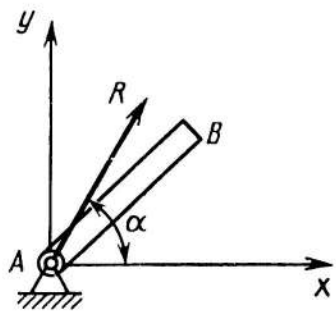

Fig. 13.
Constraint reaction in a cylindrical hinge

is the connection of two bodies by means of a bolt passed through holes in these bodies. The axial line of the bolt is called the hinge axis. In the hinged connection shown in Fig. 13, a body AB which is attached to support A can turn freely in the drawing plane around the hinge axis. In this process, the point A cannot move in directions perpendicular to the hinge axis. For that reason, the reaction $\mathbf{R}$ of the hinge may have any direction in the plane perpendicular to the hinge axis, i.e. the plane xAy. In that case, neither the magnitude of constraint reaction $\mathbf{R}$ nor its direction (angle α) is known in advance.

4. A spherical hinge or ball joint. An example of spherical hinge is a spherical support for attaching a photographic camera to a tripod. This kind of constraint

fixes a particular point of a body so that it can perform no movements in space. The reaction **R** of this kind of constraint may have any direction in space; neither the magnitude of reaction **R** nor the angles that it forms with the axes x, y and z are known in advance (Fig. 14).

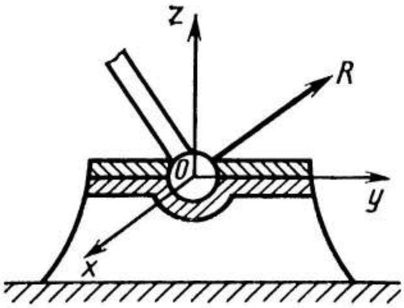

Fig. 14.
Constraint reaction in a ball joint

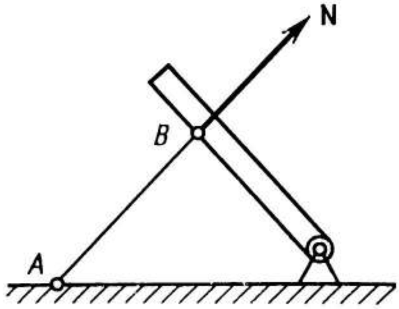

Fig. 15.
Constraint reaction in a bar

5. A bar. In some structures, a constraint can be formed by a bar hingedly fixed at the ends (Fig. 15). Assuming that the mass of the bar is negligible compared with the load onto the bar, the latter will be acted upon only by two forces applied respectively in points A and B.

For equilibrium of the bar AB, the forces applied in points A and B should, by virtue of the first axiom, be directed along the same line, i.e. along the bar axis. Therefore, a bar loaded at the ends and whose mass is negligible compared with the active load, is subject to only tension or only compression. If such a bar is used as a constraint, the constraint reaction **N** will be directed along the bar axis.

2.2. Composition and Resolution of Forces in a Plane

A system of forces whose lines of application intersect in a single point is called converging. By the corollary from the axioms of statics, a force can be transferred

along the line of its application. Proceeding from this, a system of converging forces can be reduced to a system of forces applied in that point.

The simplest case is the composition of two forces applied in the same point. If two forces, $\mathbf{F}_1$ and $\mathbf{F}_2$, are applied in point A (Fig. 16), the resultant $\mathbf{R}$ of these forces can be found on the basis of the third axiom of statics (by the rule of the parallelogram of forces). It is applied in the same point A and represented in the magnitude and direction by the diagonal of the parallelogram constructed on these forces as the sides.

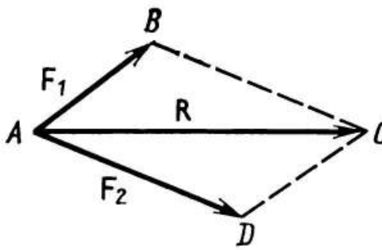

Fig. 16.
Composition of two forces

The resultant force can also be determined by constructing not the entire parallelogram $ABCD$, but only one of the triangles ABC or ADC. For constructing any one of them, say, ADC (see Fig. 16), a line DC is drawn from the end of the vector $\mathbf{F}_2$, which is equal to the vector $\mathbf{F}_1$. The closing side AC of triangle ADC represents in magnitude and direction the resultant of these two converging forces. The triangle ADC (or ABC) thus obtained is called the force triangle and the method of composition of forces, the rule of triangle.

The rule of force triangle can also be used in composition of more than two forces (Fig. 17). First, two of the given forces, say $\mathbf{F}_1$ and $\mathbf{F}_2$, are composed by this rule. A line section BC, which is equal to the force vector $\mathbf{F}_2$, is drawn from the end of vector $\mathbf{F}_1$. The closing side of the triangle ABC represents the resultant $\mathbf{R}_{1,2}$ of the forces $\mathbf{F}_1$ and $\mathbf{F}_2$. Then, using the same rule, the forces $\mathbf{R}_{1,2}$ and $\mathbf{F}_3$ are composed by drawing a line section CD equal to the force $\mathbf{F}_3$ from the point C and connecting the points A and D. The section AD thus obtained is the resultant of the forces $\mathbf{R}_{1,2}$ and $\mathbf{F}_3$, i.e. it replaces the action of the forces $\mathbf{F}_1$, $\mathbf{F}_2$ and $\mathbf{F}_3$. Continuing in the same manner, we obtain the vector $\mathbf{R} = AE$ which is the resultant of the given system of converging forces,

Composition of forces can also be carried out without constructing every time a force triangle. It suffices **to** draw the vector $\mathbf{F}_2$ from the end of vector $\mathbf{F}_1$ (point B), then the vector $\mathbf{F}_3$ from the end of vector $\mathbf{F}_2$ (point C), and so on (Fig. 18). Connecting the point A of application

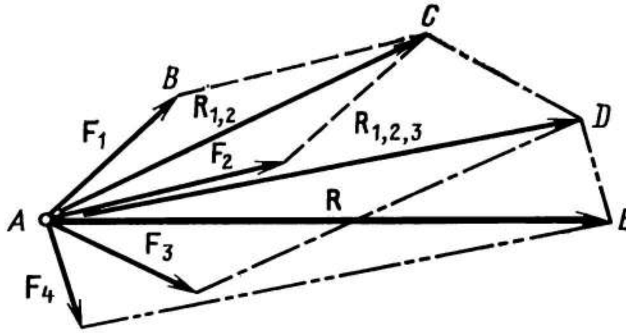

Fig. 17.
Composition of several forces

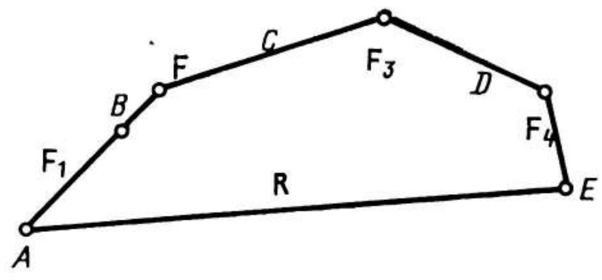

Fig. 18.
Composition of forces by the rule of polygon

of the forces with the end of vector $\mathbf{F}_4$, we obtain the same resultant $\mathbf{R}$. This method for constructing the resultant force is called the rule of polygon, the line $ABCDE$ is called the force polygon, and the line section AE is the closing line of the polygon.

In the general case of a number of forces $\mathbf{F}_1$, $\mathbf{F}_2$, . . ., . . ., $\mathbf{F}_n$ converging in the same point, the resultant of these forces, $\mathbf{R}$, is applied in the point O and equal to the geometrical sum of the force vectors:

$$\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 + \ldots + \mathbf{F}_n = \sum \mathbf{F}_k$$

The resolution of a force into components implies that the force in question is replaced by a number of forces applied in the same point and producing the same effect as the force being resolved. In other words, if the components are added geometrically, they should give the given force, i.e. the latter must be the diagonal of a parallelogram constructed on the sought-for component forces. Since an infinite number of parallelograms can be constructed on a given diagonal, certain additional data are needed for the solution of the problem, for instance, the lines of application of the components (m and n in Fig. 19). To find these components, one has to draw straight lines from the point A of the application of force **F** and from the point B of the end of that force parallel to the lines m and n. In the parallelogram $ACBD$ thus obtained, the force **F** being resolved is a diagonal and the vectors $\mathbf{AC} = \mathbf{F}_m$ and $\mathbf{AD} = \mathbf{F}_n$ applied in point A are the sought-for components:

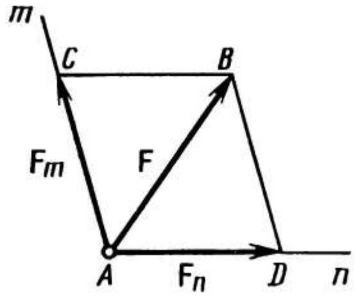

Fig. 19.
Resolution of a force into components

$$\mathbf{F} = \mathbf{F}_m + \mathbf{F}_n$$

Thus, any system of converging forces can be replaced by a resultant force. The forces applied in the same point are balanced if only their resultant force is equal to zero. Since the resultant of a system of converging forces is equal to the closing side of the force polygon constructed on these forces, an essential condition for equilibrium is that the force polygon must be closed.

On the other hand, the closure of a force polygon implies that the resultant of converging forces is equal to zero. Thus, the geometrical form of equilibrium of converging forces can be formulated as follows: for equilibrium of a system of converging forces it is required and sufficient that the force polygon constructed for that system be closed. The closed force polygon for a balanc-

ed plane system of converging forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, and $\mathbf{F}_4$ is shown in Fig. 20.

The analytical form of the equilibrium conditions can be obtained on the basis of the following theorem: the projection of a geometrical sum of vectors onto each axis is equal to the algebraic sum of the projections of component vectors onto the same axis. The projection of a force onto an axis is the section of the axis confined between two perpendiculars drawn onto the axis from the beginning and end of a force vector.

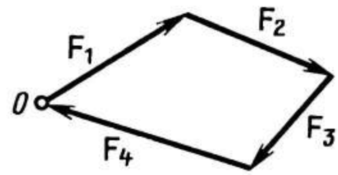

Fig. 20.
Construction of a closed polygon of forces

The geometrical sum of the vectors $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, and $\mathbf{F}_4$ (Fig. 21) is the vector $\mathbf{AE} = \mathbf{R}$ which represents the closing side of the vector polygon $ABCDE$ whose sides are

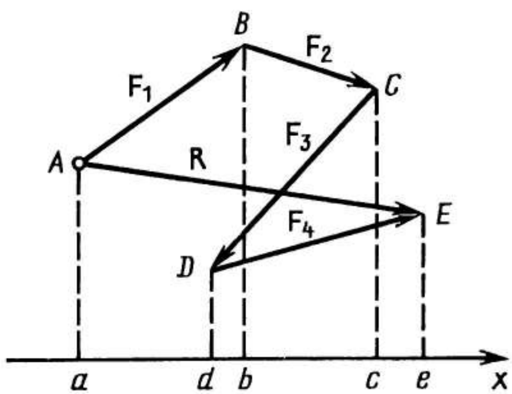

Fig. 21.
Geometrical sum of vectors

the component vectors. By projecting the vectors onto the x axis, we obtain:

$$\mathbf{F}_{1x} = ab, \ \mathbf{F}_{2x} = bc, \ \mathbf{F}_{3x} = -cd, \ \mathbf{F}_{4x} = de, \text{ and}$$
$$\mathbf{R}_x = ae.$$

As may be seen from the figure, $ae = ab + bc - cd + de$, and therefore:

$$\mathbf{R}_x = \mathbf{F}_{1x} + \mathbf{F}_{2x} + \mathbf{F}_{3x} - \mathbf{F}_{4x} = \sum \mathbf{F}_{kx}$$

as was to be proved.

Since the resultant of a system of converging forces is equal to their geometrical sum, it follows from the theorem that the projection of the resultant of a system of converging forces onto an axis is equal to the algebraic sum of the projections of component forces onto the same axis.

Denoting the projections of the resultant force onto the coordinate axes x and y as R_x and R_y and the projections of the component forces onto the same axes by X and Y with respective subscripts, we obtain:

$$R_x = X_1 + X_2 + X_3 + \ldots = \sum X_k$$
$$R_y = Y_1 + Y_2 + Y_3 + \ldots = \sum Y_k$$

The magnitude of the resultant of a plane system of converging forces can be determined by the formula (Fig. 22):

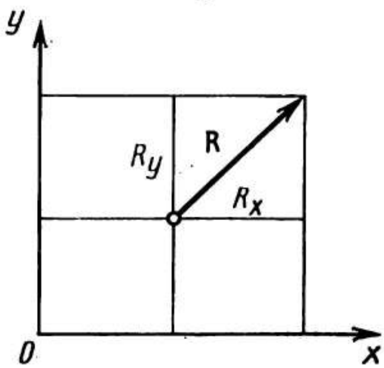

Fig. 22.
Modulus of a resultant force

$$R = \sqrt{R_x^2 + R_y^2}$$

Therefore, the magnitude of a resultant force can be determined as:

$$R = \sqrt{(\sum X_k)^2 + (\sum Y_k)^2}$$

For equilibrium, it is required that $R = 0$, and therefore, the expression under the root sign must also be zero. Since the terms under the root sign are squares of certain numbers and therefore are always positive, R can be equal to zero only if each of these terms is separately equal to zero, i.e.

$$\sum X_k = 0 \quad \text{and} \quad \sum Y_k = 0$$

The equations obtained are called the equilibrium equations and express analytically the required and sufficient conditions for equilibrium of converging forces in a plane. For equilibrium of a system of converging forces it is required and sufficient that the algebraic sum of the projection of all forces onto each of the coordinate axes be equal to zero.

2.3. Parallel Forces

Let us consider a solid body acted upon by two parallel forces $\mathbf{F}_1$ and $\mathbf{F}_2$ applied in points A and B and directed in the same sense (Fig. 23).

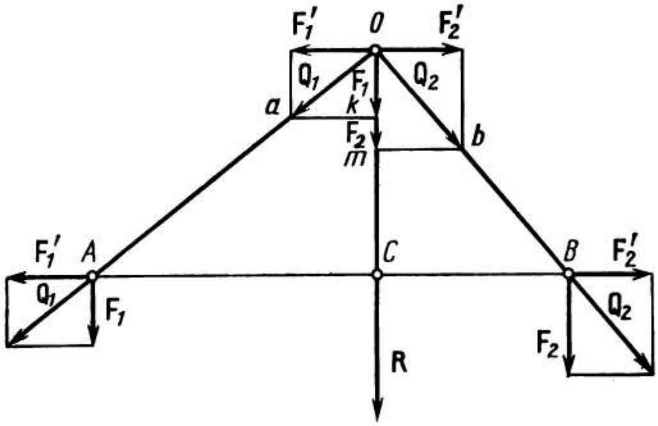

Fig. 23.
Resultant of two parallel forces directed in the same sense

The resultant of these forces will be found by using the first and second axioms of statics and by replacing the given system of parallel forces by an equivalent system of converging forces $\mathbf{Q}_1$ and $\mathbf{Q}_2$. Let two additional balanced forces $\mathbf{F}_1'$ and $\mathbf{F}_2'$ ($\mathbf{F}_1' = -\mathbf{F}_2'$) be applied in points A and B and directed along the line AB. These forces can be added with the forces $\mathbf{F}_1$ and $\mathbf{F}_2$ by the rule of parallelogram. The forces $\mathbf{Q}_1$ and $\mathbf{Q}_2$ thus obtained are then transferred into point O where their lines of application intersect. The forces $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are then resolved in point O into the initial components. This gives us two balanced forces $\mathbf{F}_1'$ and $\mathbf{F}_2'$ which can be rejected, and two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ directed along the same line. These forces are now transferred into point C and replaced by the resultant

force **R** whose magnitude is:

$$R = F_1 + F_2$$

The force obtained, **R**, is exactly the resultant of the parallel forces $\mathbf{F}_1$ and $\mathbf{F}_2$ applied respectively in points A and B. The position of the point of application of the resultant force can be found by considering the similarity of triangles OAC, Oak, OCB, and Omb:

$$\frac{AC}{OC}=\frac{F_1'}{F_1}; \quad \frac{BC}{OC}=\frac{F_2'}{F_2}$$

or

$$AC \cdot F_1 = BC \cdot F_2$$

since $F_1' = F_2'$.

Noting the property of proportion and considering that

$$BC + AC = AB$$

$$\text{and} \quad F_1 + F_2 = R$$

we obtain:

$$\frac{BC}{F_1}=\frac{AC}{F_2}=\frac{AB}{R}$$

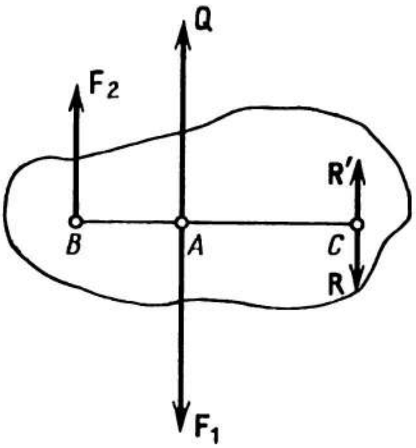

Fig. 24.
Resultant of two parallel forces directed oppositely

Thus, the resultant of two parallel forces directed in the same sense is parallel to these forces, directed in the same sense, and equal in magnitude to their sum; the line of application of the resultant force passes between the lines of application of these forces at distances inversely proportional to the magnitudes of these forces.

To find the resultant of two parallel forces of different magnitude and directed in the opposite sense (Fig. 24), we take a point C on the continuation of line BA and apply in it two additional balanced forces **R** and **R′** which are parallel to the forces $\mathbf{F}_1$ and $\mathbf{F}_2$. The magnitudes of these forces and the position of point C are chosen so as to satisfy the equalities:

$$\frac{BC}{F_1}=\frac{AC}{F_2}=\frac{AB}{R}; \quad R' = R = F_1 - F_2$$

Composition of the forces $\mathbf{F}_2$ and $\mathbf{R}'$, which are directed in the same sense, by the respective formula gives that their resultant $\mathbf{Q}$ is equal in magnitude to $F_2 + R'$, i.e. is equal to F_1 and applied in point A. The forces $\mathbf{F}_1$ and $\mathbf{Q}$ are counter-balanced and can be cancelled. The result is that the initial forces $\mathbf{F}_1$ and $\mathbf{F}_2$ are replaced by a single resultant force $\mathbf{R}$ whose magnitude and point of application are determined by the formulae given above.

Consequently, the resultant of two parallel forces directed in the opposite sense is parallel to these forces,

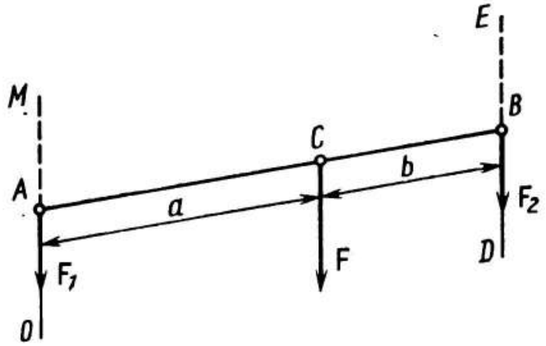

Fig. 25.
Resolution of a force into tow parallel forces

directed in the sense of the larger force, and equal in magnitude to their difference; the line of application of the resultant force passes outside the section connecting the points of application of the initial forces at distances from these points which are inversely proportional to the magnitudes of forces.

The relationships given above are suitable for solving problems on resolution of a given force into two parallel forces directed in the same or opposite sense. The problem requires certain additional data for its solution: the positions of the lines of application of both sought-for forces or the magnitude and the line of application of one of the active forces.

Consider the most common case. Suppose that a given force $\mathbf{F}$ applied in point C (Fig. 25) should be resolved into two components directed along lines OM and DE

which are parallel to the line of application of the given force. We draw an arbitrary line through point C so as to intersect the first and the second lines in points A and B. These points can be taken as the points of application of the sought-for components $\mathbf{F}_1$ and $\mathbf{F}_2$. Since the force $\mathbf{F}$ must be the resultant of the forces $\mathbf{F}_1$ and $\mathbf{F}_2$, it can be written:

$$F_1 + F_2 = F; \qquad \frac{F_1}{F_2} = \frac{CB}{AC} = \frac{b}{a}$$

Solving this system of equations, we can find the magnitudes of the sought-for forces F_1 and F_2 whose application lines are given by the distances a and b. The problem is solved in the same way for the case when the point of application of a given force $\mathbf{F}$ is not between the lines of application of two forces, but beyond one of them. In that case the magnitude of force F is equal to the difference of magnitudes of the component forces which are directed in the opposite sense.

2.4. Moment of a Force Relative to an Arbitrary Point. A Couple of Forces. The Property of Force Ccuples

The moment of a force $\mathbf{F}$ relative to an arbitrary point (centre) O is the product of the magnitude of force $\mathbf{F}$ by the length of the force arm, taken with a proper sign. The arm of force $\mathbf{F}$ relative to centre O is the perpendicular drawn from the centre O onto the line of application of force $\mathbf{F}$ (Fig. 26). Therefore,

$$m_0(\mathbf{F}) = Fh$$

Let us agree that a moment has a 'plus' sign if the force tends to turn the body around the centre O counterclockwise, and a 'minus' sign if it tends to turn the body clockwise.

The moment of a force can be characterized by the following properties:

(1) the moment of a force is not changed if the point of force application is transferred along the application line;

(2) the moment of a force relative to a centre O is equal to zero if only the force equals zero in magnitude or the arm of force is equal to zero (i.e. the line of force application passes through the centre); and

(3) the moment of a force is numerically equal to the doubled area of triangle OAB (Fig. 26):

$$m_0(\mathbf{F}) = 2S_{\Delta OAB}$$

This result follows from the fact that

$$S_{\Delta OAB} = \frac{1}{2} AB \cdot h = \frac{1}{2} Fh$$

The moment of the resultant of a plane system of converging forces relative to any centre is equal to the algebraic sum of the moments of the component forces

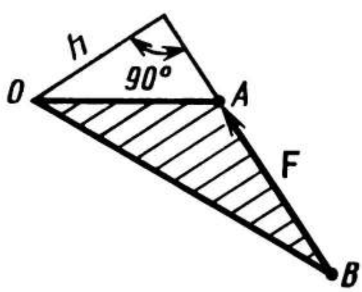

Fig. 26.
Moment of a force relative to a point

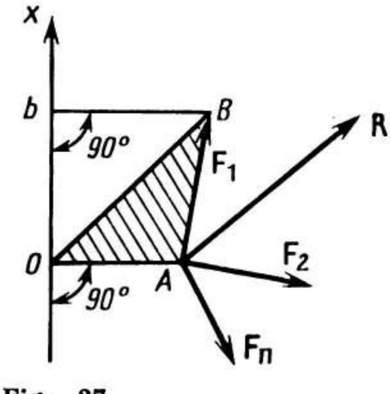

Fig. 27.
Moment of the resultant of a plane system of convergent forces

relative to the same centre. This is what is called Varignon's theorem which can be proved in the following way.

Let there be a system of forces $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_n$ converging in a point A (Fig. 27). We choose an arbitrary centre O and draw through it an axis Ox perpendicular to line OA. The positive direction of the axis Ox is selected so that the sign of the projection of any of the forces onto this axis be coincident with the sign of the moment of that force relative to centre O. Using the formula given

above, we can find the expression for the moments $m_0(\mathbf{F}_1)$, $m_0(\mathbf{F}_2)$, . . ., $m_0(\mathbf{F}_n)$. For the force $\mathbf{F}_1$ we have:

$$m_O(\mathbf{F}_1) = 2S_{\Delta OAB}$$

but, as may be seen from Fig. 27, $2S_{\Delta OAB} = Oa \cdot Ob = OA \cdot F_{1x}$, where F_{1x} is the projection of force $\mathbf{F}_1$ onto the axis Ox, and therefore:

$$m_0(\mathbf{F}_1) = OA \cdot F_{1x}$$

The moments of all other forces are determined in the same way.

Let us denote by $\mathbf{R}$ the resultant of forces $\mathbf{F}_1$, $\mathbf{F}_2$, . . ., . . ., $\mathbf{F}_n$, i.e. $\mathbf{R} = \sum \mathbf{F}_k$. According to the theorem of the projection of a sum of forces onto an axis, it may be written:

$$R_x = \sum F_{kx}$$

Multiplying both parts of this equation by OA, we get

$$OA \cdot R_x = \sum (OA \cdot F_{kx})$$

whence

$$m_O(\mathbf{R}) = \sum m_O(\mathbf{F}_k)$$

This formula is the mathematical expression of Varignon's theorem.

A particular case of parallel forces is that in which two forces are equal in magnitude and directed in the opposite sense to each other. This system of forces has a peculiar property: it has no resultant force and is not in equilibrium. Such a system of two forces equal in magnitude and directed oppositely is called a couple of forces, or simply a couple.

The plane passing through the lines of application of a couple of forces is called the plane of couple. The distance h between the lines of application of forces in a couple (Fig. 28) is called the arm of couple. A force couple acting on a body tends to rotate it if this is not prevented by the body's constraints. The rotary effect of

a couple depends on the magnitude of its forces and the length of the arm of action of these forces. In order to characterize this effect, use is made of the concept of the moment of a couple which is equal numerically to the product of the magnitude of one force by its arm:

$$m = F \cdot h$$

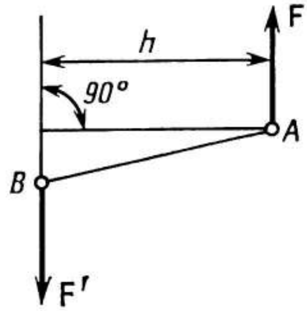

Fig. 28.
A couple of forces

It is agreed that the moment of a couple of forces is positive when the couple tends to rotate a body counter-clockwise and is negative when the rotation is clockwise. By definition of the moment of a couple, its unit must be a unit of force multiplied by a unit of length. If the force is measured in newtons and the arm in metres, the unit of the moment is a newton-metre (N m).

The properties of couples of forces are described by the three theorems as follows:

Theorem 1. The algebraic sum of the moments of forces of a couple relative to any centre lying in the plane of couple does not depend on selection of that centre and is equal to the moment of the couple. Indeed, considering the moments of the forces **F** and **F′** in a couple relative to point O (Fig. 29), we find that

$$m_O(\mathbf{F}) = -F \cdot Oa$$
$$m_O(\mathbf{F}') = F' \cdot Ob$$

Adding these equalities termwise and noting that $F = F'$ and $Ob - Oa = h$, where h is the couple arm, we obtain

$$m_O(\mathbf{F}) + m_O(\mathbf{F}') = m$$

Theorem 2. The effect produced on a body is not changed if a couple of forces applied to an absolutely rigid body is replaced by any other couple lying in the same plane and having the same moment. Let a body be acted upon by a couple of forces $\mathbf{F}_1$, $\mathbf{F}_1'$ with the arm

$AB = h$ (Fig. 30). Let us now apply in points A and B two additional forces $\mathbf{T}_1$ and $\mathbf{T}_1'$ which are equal in magnitude to each other and directed along the same line oppositely. Let the forces $\mathbf{F}_1$ and $\mathbf{T}_1$ and $\mathbf{F}_1'$ and $\mathbf{T}_1'$ be composed pairwise. The resultants $\mathbf{R}_1$ and $\mathbf{R}_1'$ constitute

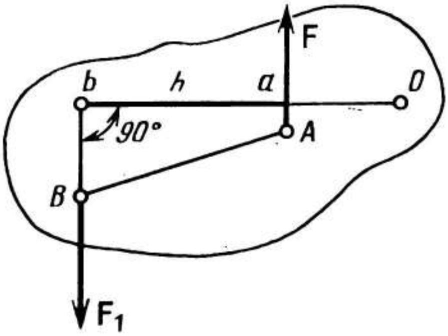

Fig. 29.
Moment of a force couple relative to an arbitrary centre

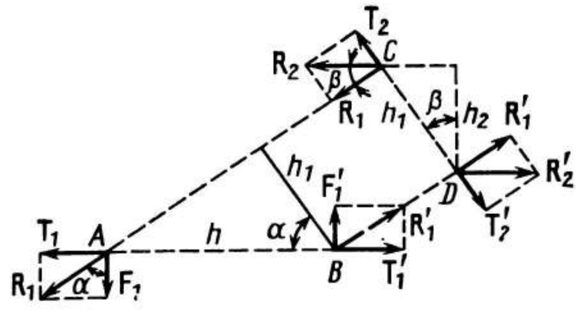

Fig. 30.
Replacement of a force couple

a new couple and, since the system of forces $\mathbf{T}_1$, $\mathbf{T}_1'$ is balanced, the couple $\mathbf{R}_1$, $\mathbf{R}_1'$ is equivalent to the given couple.

If the forces of this couple are transferred to any other points lying on the lines of their application, we will get similarly a couple $\mathbf{R}_2$, $\mathbf{R}_2'$ which is also equivalent to the given couple. All equivalent couples obtained in this way have the same sense of rotation and the same magnitude of the moment. As may be seen from Fig. 30

$$R_1 = \frac{F_1}{\cos\alpha}; \qquad h_1 = h\cos\alpha$$

and therefore

$$R_1h_1 = F_1h; \qquad R_2 = \frac{R_1}{\cos\beta}$$
$$h_2 = h_1 \cos\beta; \qquad R_2h_2 = R_1h_1$$

It follows from Theorem 2 that:

(a) a given couple can be transferred into any points in its plane without changing the effect of this couple on a body, and

(b) the magnitudes of forces or the length of arm in a couple can be changed arbitrarily without changing the effect of the couple on a body, the moment of the couple being kept unchanged.

It should be emphasized that the principle of transfer of a couple in its plane is applicable only to absolutely rigid bodies and is utilized for solving problems of theoretical mechanics. When solving problems of strength of materials, the points of application of a couple of forces must be additionally specified. Since the effect of a couple of forces on a body is determined completely by its moment, the moment of a couple, rather than the product of a force by its arm, is usually given in problems on mechanics. In that case, a couple of forces is often symbolized by a circular arrow which indicates the direction of rotation, while the forces proper are not shown. The magnitude of the moment is written at the arrow.

Theorem 3. A system of couples lying in the same plane is equivalent to a single couple lying in the same plane and having the moment equal to the algebraic sum of the moments of component couples.

Let three couples of forces with the moments m_1, m_2, and m_3 be acting on a body (Fig. 31). Using Theorem 2 on the equivalence of couples, the given couples can be replaced by three couples $(\mathbf{F}_1, \mathbf{F}_1')$, $(\mathbf{F}_2, \mathbf{F}_2')$, and $(\mathbf{F}_3, \mathbf{F}_3')$ having a common arm h and the same moments:

$$F_1h = m_1; \quad -F_2h = m_2; \quad \text{and} \quad F_3h = m_3$$

The composition of the individual forces applied in points A and B gives a force $\mathbf{R}$ applied in point A and

force $\mathbf{R}'$ applied in point B, which are equal to each other in magnitude:

$$R = R' = F_1 - F_2 + F_3$$

Thus, the entire system of couples will be replaced by a single couple $\mathbf{R}$, $\mathbf{R}'$ with the moment

$$M = Rh = F_1h + (-F_2h) + F_3h = m_1 + m_2 + m_3$$

The same result will evidently be obtained for any number of couples of forces. A system of n couples with

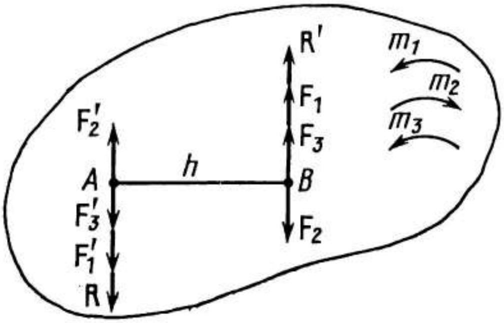

Fig. 31.
Replacement of a system of force couples by a single force couple

the moments m_1, m_2, m_3, etc. can therefore be replaced by a single couple with the moment

$$M = \sum m_k$$

2.5. Planar System of Arbitrarily Arranged Forces

A planar system of forces arranged arbitrarily in a plane can be reduced to a single centre by using the following theorem: a force applied to an absolutely rigid body can be transferred parallel to itself, without changing the effect of the force on the body, into an arbitrary point of the body if a couple of forces is added with the moment equal to the moment of the force transferred relative to the new point of force application.

To prove the theorem, consider a body acted upon by a force **F** applied in point A (Fig. 32). If two balanced forces **F**$'$ and **F**$''$ are applied in any other point B such that $\mathbf{F}' = \mathbf{F}$ and $\mathbf{F}'' = -\mathbf{F}$, the effect of force **F** on the body will not be changed. The system of three forces thus obtained represents the force **F**$'$ which is equal to **F**, but applied in point B and couple of forces $(\mathbf{F}, \mathbf{F}'')$ with the moment

$$m = m_B(F)$$

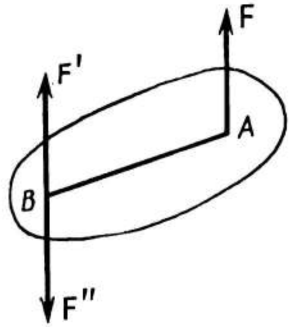

Fig. 32.
Transfer of a force

Using this theorem, all forces $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_n$ acting on the body can be transferred into a single arbitrary point O (Fig. 33*a*) which is called the reduction centre. After that, the body will be acted upon by a system of forces $\mathbf{F}_1 = \mathbf{F}_1', \mathbf{F}_2 = \mathbf{F}_2', \ldots, \mathbf{F}_n = \mathbf{F}_n'$ applied in the centre O and a system of couples whose moments are:

$$m_1 = m_O(\mathbf{F}_1), \quad m_2 = m_O(\mathbf{F}_2), \quad \ldots, \quad m_n = m_O(\mathbf{F}_n)$$

The forces applied in centre O can be replaced by a single force **R** applied in the same centre and equal to

$$\mathbf{R} = \sum (\mathbf{F}_k')$$

or, noting that $F_k' = F_k$,

$$\mathbf{R} = \sum \mathbf{F}_k$$

Similarly, all couples can be replaced by a single couple lying in the same plane whose moment

$$M_O = \sum m_O(\mathbf{F}_k)$$

The vector **R**, which is equal to the geometrical sum of all forces of the given system, is called the main vector of a system; the moment M_0, which is the sum of the moments of all forces of the system relative to a point, is called the main moment of that system relative to that point.

Thus, any planar system of forces acting on an absolutely rigid body, when reduced to an arbitrary centre O, is replaced by a single force **R** equal to the main vector of the system and applied in the reduction centre O, and a single couple with the moment M_0 equal to the main moment of the system relative to the centre O (Fig. 33*b*).

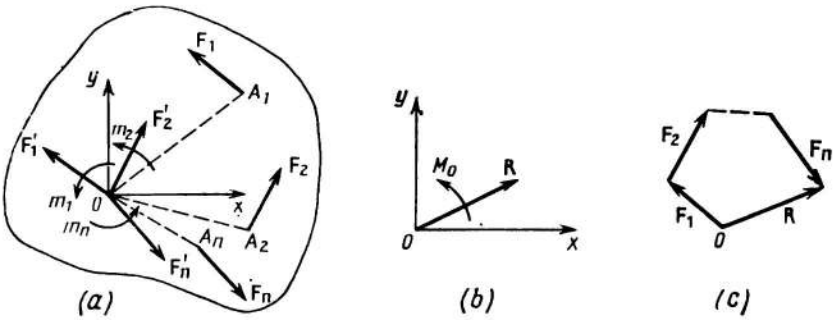

Fig. 33.
Reduction of an arbitrary planar system of forces to a single centre

It is clear that two systems of forces which have the same main vectors and main moments are statically equivalent, and therefore, in order to specify a planar system of forces, it suffices to specify its main vector **R** and the main moment M relative to a certain centre.

It should be noted that the force **R** cannot be regarded the resultant of a given system of forces, since it replaces the system only together with a couple of forces. The magnitude of **R** can be found either by geometrical construction of a force polygon (Fig. 33*c*) or analytically by the formulae:

$$R_x = \sum F_{kx}; \quad R_y = \sum F_{ky}; \quad R = \sqrt{R_x^2 + R_y^2}$$

where F_{kx} and F_{ky} are the projections of the forces onto corresponding axes of coordinates.

The magnitude of R does not depend on the choice of centre O. The magnitude of M is determined by the location of centre O, because of which it is essential to indicate the centre relative to which the main moment has been calculated.

For equilibrium of a system of forces arranged arbitrarily in a plane, it is required and sufficient that the main vector and the main moment relative to an arbitrarily chosen reduction centre be equal to zero:

$$R = 0; \qquad M = 0$$

These conditions are indispensable, since if one of them is not observed, the system of active forces will be reduced to a resultant force ($R \neq 0$) or to a couple ($M_0 \neq 0$), and therefore, is not balanced. On the other hand, these conditions are sufficient, since, with $R = 0$, the system can be reduced only to a couple with the moment M_0 and, since $M_0 = 0$, it will be balanced.

Let us now consider various forms of analytical conditions for equilibrium.

1. As follows from the relationships given above, R can be equal to zero only when $R_x = 0$ and $R_y = 0$, and M_0 can be equal to zero only when $\Sigma m_0 (\mathbf{F}_k) = 0$.

Therefore, it is possible to write the following analytical conditions of equilibrium:

$$\sum F_{kx} = 0, \quad \sum F_{ky} = 0, \quad \text{and} \quad \sum m_O (\mathbf{F}_k) = 0$$

For equilibrium of a planar system of arbitrarily arranged forces, it is required and sufficient that the sums of projections of all forces onto each of two arbitrary coordinate axes lying in the plane of application of the forces and the sum of algebraic magnitudes of the moments of all forces relative to any point in that plane be equal to zero.

2. For equilibrium of a system of forces arranged arbitrarily in a plane, it is required and sufficient that the sum of the moments of all these forces relative to any centres A and B and the sum of their projections onto the Ox axis not perpendicular to line AB be equal to zero:

$$\sum m_A (\mathbf{F}_k) = 0, \quad \sum m_B (\mathbf{F}_k) = 0, \quad \sum \mathbf{F}_{kx} = 0$$

The necessity of these conditions is quite clear, since if anyone of them is not observed, then either $R \neq 0$ or $M \neq 0$ and there will be no equilibrium. The sufficiency of these conditions follows from the following reasoning.

If there existed a resultant force, the line of its application should pass through the points A and B. In that case, however, the projection of the resultant $R_x = \Sigma F'_{kx}$ onto the axis x, which is not perpendicular to line AB, cannot be equal to zero.

3. For equilibrium of any arbitrary planar system of forces, it is required and sufficient that the sums of the moments of all these forces relative to any three centres A, B, and C, not lying on the same straight line, be equal to zero:

$$\sum m_A(\mathbf{F}_k) = 0, \quad \sum m_B(\mathbf{F}_k) = 0, \quad \sum m_C(\mathbf{F}_k) = 0$$

The necessity of these conditions is also quite clear, since if anyone of them is not observed, then either $m_A(\mathbf{F}_k) \neq 0$ or $m_B(\mathbf{F}_k) \neq 0$ or else $m_C(\mathbf{F}_{k'}) \neq 0$ and there will be no equilibrium.

The sufficiency of these conditions is proved by the following reasoning. If, with observance of all these conditions, a given system forces were not in equilibrium, it would be possible to reduce that system to a resultant passing simultaneously through points A, B, and C, which is impossible, since these points do not belong to the same straight line. Therefore, observance of these conditions involves equilibrium of the system.

From all the cases considered, the equations of conditions 1 are the principal conditions of equilibrium, since they impose no restrictions on selection of coordinate axes and the centre of moment. If all forces acting on a body are parallel to one another, then the axis Ox can be directed perpendicular to these forces and the axis Oy, parallel to them (Fig. 34). Then the projections of all forces onto the Ox axes will be equal to zero and the conditions 1 will be reduced to two mere equations of equilibrium

$$\sum \mathbf{F}_{ky} = 0 \quad \text{and} \quad \sum m_0(\mathbf{F}_k) = 0$$

Using the equations of the conditions 2, it is possible to obtain another form of equilibrium conditions for parallel forces:

$$\sum m_A(\mathbf{F}_k) = 0 \quad \text{and} \quad \sum m_B(\mathbf{F}_k) = 0$$

provided that points A and B do not lie on a straight line parallel to the forces.

When making a choice between various forms of equilibrium conditions, the form giving the simplest system of equations should be preferred. In the simplest system of equations, each equation contains only one

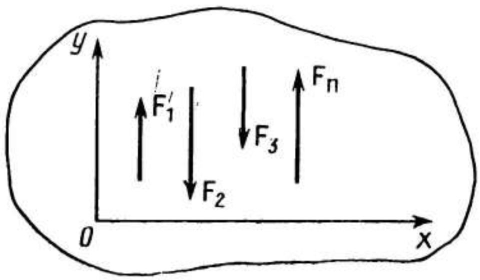

Fig. 34.
Equilibrium conditions for the action of parallel forces

unknown. The following recommendations are essential for finding simpler equations:

(a) when compiling the equations of projections, the coordinate axis should preferably be drawn perpendicular to an unknown force;

(b) when compiling the equations of moments, the centre of moments should be preferably located in a point where a larger number of unknown forces intersect.

In calculations of moments it is possible, by using Varignon's theorem, to resolve a given force into two components and to find the moment as the sum of the moments of these components. These equilibrium conditions are used for determining the reactions of supports, i.e. elements of attachment of various structures, beams, trusses etc.

The most typical kinds of support fastenings are as follows:

(1) a movable hinged support (Fig. 35); the reaction of a movable support A is directed along the normal to the surface against which the rollers of the movable support bear;

(2) a fixed hinged support (see Fig. 35); the reaction $\mathbf{R}_B$ of a fixed support B passes through the axis of the

hinge and can have any direction in the plane of drawing; in calculations, the reaction $\mathbf{R}_B$ is usually replaced by components $\mathbf{X}_B$ and $\mathbf{Y}_B$ directed along the coordinate

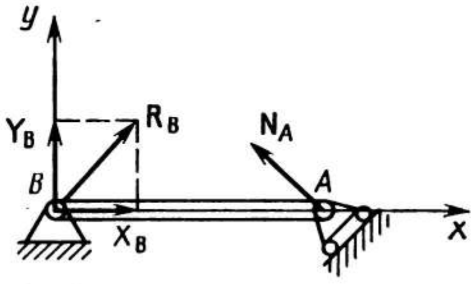

Fig. 35.
Hinged supports

axes; when X_B and Y_B are found, the magnitude of reaction $\mathbf{R}_B$ is determined by the equation:

$$R_B = \sqrt{X_B^2 + Y_B^2}$$

(3) fixed (built-in) anchorage (Fig. 36); this type of support permits neither linear displacements nor turning of a beam. In that case the support surfaces develop a system of distributed reaction forces which act on the built-in end of the beam. This system includes an unknown reaction force $\mathbf{R}_A$ applied in that point and a couple of forces with an unknown moment M_A. The force $\mathbf{R}_A$ can in turn be resolved into components $\mathbf{X}_A$ and $\mathbf{Y}_A$.

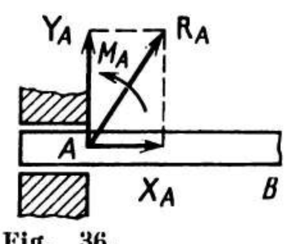

Fig. 36.
Fixed attachment of a beam end

Thus, the problem of determination of the reaction of a built-in support reduces to determining three unknowns: the components $\mathbf{X}_A$ and $\mathbf{Y}_A$ which restrict any linear displacement of a beam in the plane of application of forces and the moment M_A which prevents twisting of the beam under the action of the forces applied.

2.6. Arbitrary Three-dimensional System of Forces

A system of forces whose lines of application lie in different plane is called three-dimensional.

A three-dimensional system of forces is termed converging if the lines of application of all forces intersect in a single point. By analogy with a planar system of forces, such a system can be reduced to a system of forces applied in one point.

Let four forces: $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, and $\mathbf{F}_4$ not lying in on plane be applied in point O (Fig. 37). Since a plane can always be drawn through any two intersecting straight lines, every two of the forces considered necessarily belong to a single plane. For composition of these pairs of forces (say of $\mathbf{F}_1$ and $\mathbf{F}_2$), use may be made of the rule of composition of converging forces in a plane, which enables us to find their resultant $\mathbf{R}_1$ applied in the same point O. Composition of this resultant with another force say, $\mathbf{F}_3$, gives the resultant of three forces:

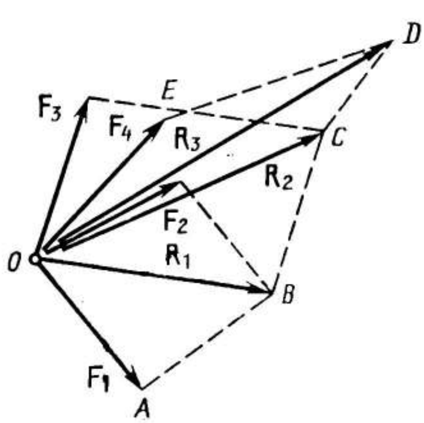

Fig. 37.
Resultant ~~of a~~ spatial system of forces

$$\mathbf{R}_2 = \mathbf{R}_1 + \mathbf{F}_3 = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3$$

Having made the same procedure for $\mathbf{R}_2$ and $\mathbf{F}_4$, we finally obtain a force vector $\mathbf{R}_3$.

To find the resultant of a system of n forces, the procedure should be repeated $n - 1$ times. The resultant force will then be equal to the vector sum of the forces considered:

$$\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \ldots + \mathbf{F}_n$$

or

$$\mathbf{R} = \sum \mathbf{F}_k$$

As may be seen from Fig. 37, the resultant of a three-dimensional system of converging forces is represented graphically in its magnitude and direction by the closing side of the polygon *OABCD* constructed on the component forces.

It should be emphasized that the force polygon of a three-dimensional system of forces does not lie in

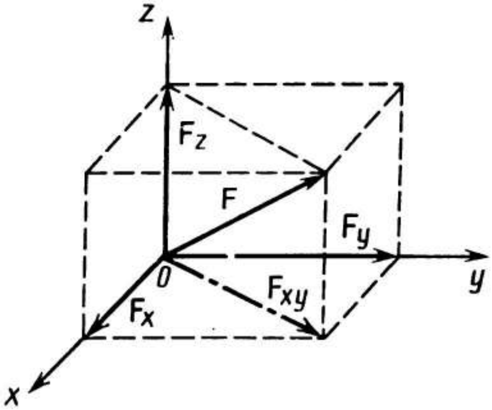

Fig. 38.
Resolution of the resultant of a spatial system of forces into three mutually perpendicular directions

a single plane, because of which the resultant is found analytically, rather than graphically.

Suppose that a force **F** is given (Fig. 38). We select a system of coordinates so that the origin of coordinates is in the beginning of the force vector **F**.

A perpendicular is then drawn from the end of this vector onto the xy plane and the force **F** is resolved into components $\mathbf{F}_{xy}$ and $\mathbf{F}_z$. The component $\mathbf{F}_{xy}$ can in turn be resolved into components $\mathbf{F}_x$ and $\mathbf{F}_y$. It can then be written:

$$\mathbf{F} = \mathbf{F}_x + \mathbf{F}_y + \mathbf{F}_z$$

Having complemented the graphical construction to a parallelepiped, it is possible to formulate the following conclusion: the resultant of three mutually perpendicular forces $\mathbf{F}_x$, $\mathbf{F}_y$, and $\mathbf{F}_z$ is represented in the magnitude and direction by the diagonal of the parallelepiped constructed on the forces.

As may be seen from Fig. 38, when the force **F** is resolved into three mutually perpendicular directions x, y, z, its components are equal to the projections of force **F** onto these axes. These projections are designated respectively F_x, F_y, and F_z.

When the projections of a force onto three mutually perpendicular axes of coordinates are known, the magnitude and direction of the force vector can be determined by the formulae:

the magnitude of force

$$\mathbf{F} = \sqrt{\mathbf{F}_x^2 + \mathbf{F}_y^2 + \mathbf{F}_z^2}$$

and the direction cosines:

$$\cos(\mathbf{F},\ x) = \frac{F_x}{F}; \quad \cos(\mathbf{F},\ y) = \frac{F_y}{F}; \quad \cos(\mathbf{F},\ z) = \frac{F_z}{F}$$

As is proved in more advanced courses in theoretical mechanics, the projection of a geometrical sum of vectors onto any axis is equal to the algebraic sum of the projections of component vectors onto the same axis. This statement, which is equally valid for planar and three-dimensional vector polygons, suggests that the projection of the resultant of a system of converging forces onto an axis is equal to the algebraic sum of the projections of the component forces onto that axis:

$$R_x = \sum X_k = F_{1x} + F_{2x} + F_{3x} + \ldots + F_{nx}$$

$$R_y = \sum Y_k = F_{1y} + F_{2y} + F_{3y} + \ldots + F_{ny}$$

$$R_z = \sum Z_k = F_{1z} + F_{2z} + F_{3z} + \ldots + F_{nz}$$

When the projections R_x, R_y, R_z of the resultant of a system of forces are known, the magnitude and direction of the resultant vector can be found by the formulae:

$$R = \sqrt{R_x^2 + R_y^2 + R_z^2}$$

$$\cos(\mathbf{R},\ x) = \frac{R_x}{R}; \quad \cos(\mathbf{R},\ y) = \frac{R_y}{R}; \quad \cos(\mathbf{R},\ z) = \frac{R_z}{R}$$

If a three-dimensional system of converging forces is in equilibrium, the resultant of this system is equal to

zero, and therefore, the projections R_x, R_y, R_z of the resultant are also equal to zero.

Hence follow the following conditions for equilibrium of a three-dimensional system of converging forces: for a three-dimensional system of converging forces to be in equilibrium, it is required and sufficient that the algebraic sum of the projections of all forces onto each of three coordinate axes be equal to zero.

Before discussing the equilibrium conditions for a three-dimensional system of arbitrarily arranged forces, let us

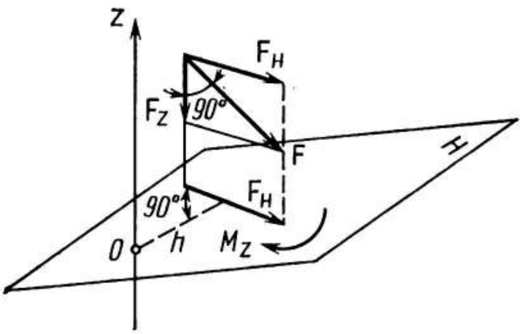

Fig. 39.
Moment of a force relative to an axis

introduce the concept of the moment of a force relative to an axis.

The moment of a force relative to an axis is defined as the moment of the projection of that force onto a plane perpendicular to that axis, taken relative to the point of intersection of the axis and plane (Fig. 39): $M_z(F) = F_H h$.

Let us agree that the moment is positive if, when looking at the positive end of the axis, the force tends to cause counter-clockwise rotation, and vice versa.

If the force is transferred along its line of application, the moment of that force relative to the axis is not changed.

The moment of a force relative to the axis is equal to zero in the following cases:

(a) when the force vector is parallel to the axis, since the projection of the force onto a plane perpendicular to the axis is then equal to zero (force $\mathbf{F}_z$ in Fig. 39);

(b) when the line of application of a force intersects the axis (and the arm of the force is then equal to zero).

The conditions of equilibrium for a three-dimensional system of arbitrary forces will be given without proof, since their strict substantiation is beyond the scope of the present course. For equilibrium of a three-dimensional system of arbitrarily arranged forces it is required and sufficient that the algebraic sum of the projections of all forces onto each of the coordinate axes be equal to zero and that the algebraic sum of the moments of all forces relative to each of these axes be also equal to zero:

$$\begin{aligned} \sum X_k &= 0, & \sum M_x(\mathbf{F}_k) &= 0; \\ \sum Y_k &= 0, & \sum M_y(\mathbf{F}_k) &= 0; \\ \sum Z_k &= 0, & \sum M_z(\mathbf{F}_k) &= 0 \end{aligned}$$

In cases when a three-dimensional system of parallel forces (for instance, parallel to the z axis) acts on a body, the equilibrium conditions for the system are written as:

$$\sum Z_k = 0, \quad \sum M_x(\mathbf{F}_k) = 0, \quad \sum M_y(\mathbf{F}_k) = 0$$

The remaining three equilibrium conditions will then reduce to identities, since the projections of all forces of such a system onto the axes x and y are equal to zero and the moments of all forces relative to the z axis are also equal to zero.

2.7. Worked Problems on Statics

The following general recommendations will be useful for solving the examples given in this section:

(1) select a solid body whose equilibrium must be studied for determining the sought-for quantities;

(2) present graphically the forces applied to the body;

(3) reject mentally the constraints imposed on the body and replace them by corresponding reaction forces;

(4) select a system of Cartesian coordinates which should preferably be oriented so as to be parallel or perpendicular to the greatest number of unknown forces and so that the lines of application of these forces intersect these axes;

(5) set up the equilibrium equations for the body;

(6) solve the system of equations and find the unknowns.

Example 1. Four forces lying in the same plane are applied to a body in point O. The first force is directed

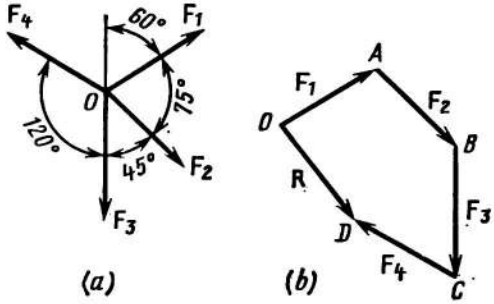

Fig. 40.
To example 1

upwards at an angle of 60° from the vertical and the other angles between the forces are 75°, 45°, and 120° (Fig. 40*a*). The magnitudes of forces are: $F_1 = 1500$ N, $F_2 = 2000$ N, $F_3 = 2500$ N, and $F_4 = 1200$ N.

Determine by graphical construction the magnitude and direction of the resultant force.

Solution. We choose a scale of forces, say, $\mu = 100$ N/mm and lay off straight sections corresponding to the magnitudes of the forces along the specified directions (Fig. 40*b*). A section OA representing the given force vector $\mathbf{F}_1$ is first laid off from an arbitrary point O. Then a section AB which represents the force $\mathbf{F}_2$ is laid off from the end of that section, i.e. from point A, and so on.

The closing side OD of the force polygon thus obtained will represent graphically on the chosen scale the resultant of the given system of forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, and $\mathbf{F}_4$ both in magnitude and direction. The length of the closing side OD is 30 mm, and therefore, the magnitude

of the resultant force $R = \mu \cdot OD = 100 \times 30 = 3000$ N. Having measured the angle between the vertical and the direction of the closing side, we find that the resultant force is directed downwards and to the right at an angle of 32° from the vertical.

Example 2. A crane shown schematically in Fig. 41 has a chain $AB = 1$ m and a brace $CB = 1.5$ m which are fastened respectively in points A and C. The distance

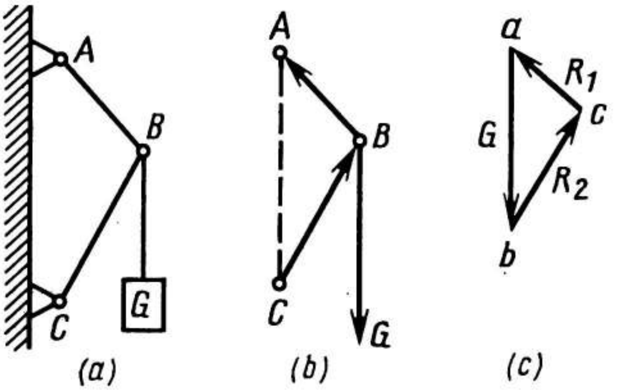

Fig. 41.
To example 2

between the points A and C is 2.5 m. A body of a weight $G = 20$ kN is suspended from point B. It is required to determine the forces in the chain AB and brace BC.

Solution. Consider the equilibrium of the three forces applied in point B: weight G and reactions of the chain and brace, R_1 and R_2. The lines of application of the reaction forces coincide with the directions of sections AB and BC (Fig. 41*b*).

We construct a closed triangle of these forces, *abc* (Fig. 41*c*), from which it is possible to determine the directions of reactions R_1 and R_2. Applying these reactions to point B, we can find that the brace BC is compressed and the chain AB tensioned. As follows from the similarity of triangles *abc* and ABC,

$$\frac{G}{AC} = \frac{R_1}{AB} = \frac{R_2}{CB}$$

These equalities give us the sought-for forces:

$$S_1 = G\frac{AB}{AC} = 20\frac{1}{2.5} = 8\ \text{kN}$$

$$S_2 = G\frac{CB}{AC} = 20\frac{1.5}{2.5} = 12\ \text{kN}$$

Example 3. A homogeneous bar AB of weight G_1 rests at one end on a horizontal plane and at the other, on a vertical plane (Fig. 42). A thread is attached to the end A of the bar, passed over a pulley C and loaded by a weight G_2.

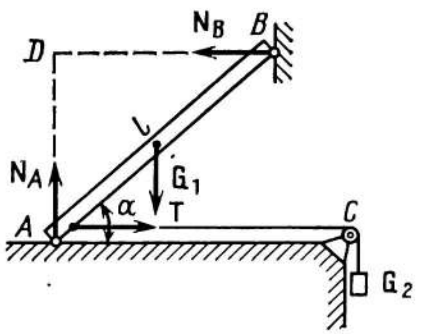

Fig. 42.
To example 3

The length of the bar is $AB = l$. Determine the angle α between the horizontal plane and bar under equilibrium conditions. Friction in pulley C can be neglected.

Solution. Consider the equilibrium of the bar. We know the force of weight G_1 of the body, which is applied in the mid of the bar and the tension of the thread, which can be denoted by T.

The reactions N_A and N_B are directed perpendicular to the supporting plane surfaces. We can now write the equilibrium equation, taking the sum of moments relative to the point of intersection of the application lines of reactions N_A and N_B

$$\sum m_D(F_k) = -G_1\frac{l}{2}\cos\alpha + G_2 l\sin\alpha = 0$$

from which the sought-for angle α can be found:

$$\tan\alpha = \frac{G_1}{2G_2}$$

Example 4. A homogeneous bar AB of a weight $G_1 =$ 200 N rests at one end on a smooth horizontal floor and at the other, on a smooth plane inclined at 45° to the

horizontal (Fig. 43). The bar is held at the end B by a cable passed over a pulley C and carrying a load of weight G_2. The strand BC of the cable is parallel to the inclined plane.

Determine the weight G_2 and the reactions of the floor and inclined plane, $\mathbf{N}_A$ and $\mathbf{N}_B$. Friction in the pulley is neglected.

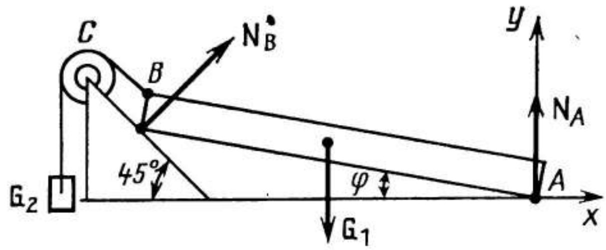

Fig. 43.
To example 4

Solution. Consider the equilibrium of the bar AB. The force of weight G_1 is known; it is applied in the mid of bar AB and directed vertically downwards. The unknown forces are the reactions $\mathbf{N}_A$ and $\mathbf{N}_B$ and the weight G_2. Since the horizontal and inclined planes are smooth, the reactions $\mathbf{N}_A$ and $\mathbf{N}_B$ are directed perpendicularly to them. The force of weight G_2 acts along the cable BC. All forces are in the same plane xy.

The chosen direction of coordinate axes are shown in the figure. We have to set up the equilibrium equation. The sum of the projections of all forces onto the x axis is:

$$\sum x = N_B \cos 45° + G_2 \cos 135° = 0$$

and the sum of their projections onto the y axis is:

$$\sum y = -G_1 + N_A + N_B \cos 45° + G_2 \cos 45° = 0$$

The sum of the moments of all forces relative to point B is:

$$\sum m_B (F_k) = -G_1 \frac{AB}{2} \cos \varphi + N_A AB \cos \varphi = 0$$

We find from the first equation that $N_B = G_2$, and from the third that $N_A = G_1/2 = 100$ N. Substituting $G_2 = N_B$ into the second equation, we obtain:

$$2N_B \cos 45^\circ = G_1 - N_A$$

whence

$$N_B = G_2 = \frac{100}{2\cos 45^\circ} = 70.7 \text{ N}$$

Example 5. A load of weight $G_1 = 500$ N is held in equilibrium by two ropes: rope BC fastened to the ceiling and making an angle of 45° with the horizontal and rope AB passed over a fixed pulley A and carrying a weight G_2 (Fig. 44). The rope AB makes an angle of 60° with the horizontal.

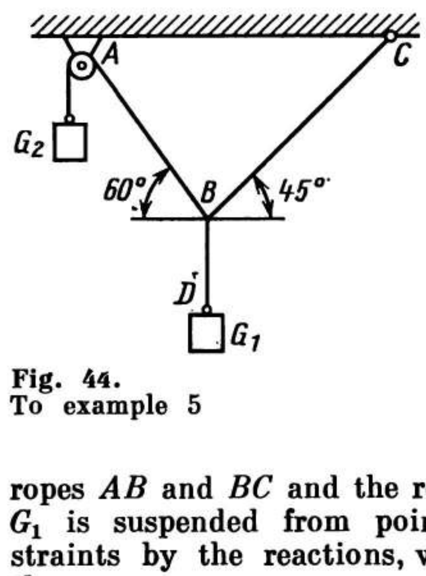

Fig. 44.
To example 5

Find the weight G_2 and the tension of rope BC. Friction in the pulley A can be neglected.

Solution. Consider the equilibrium for the point B. In order to remove the constraints, we cut mentally the ropes AB and BC and the rope BD by which the weight G_1 is suspended from point B, and replace their constraints by the reactions, which are equal to tensions of the ropes.

The point B is in equilibrium under the actions of three forces (Fig. 45): the reaction T_{BC} of rope BC, the reaction of rope BD which is equal to T_{BD}, and the reaction of rope AB, i.e. T_{AB}.

Choosing the coordinate axes as shown in Fig. 45, we can set up two equations for equilibrium of the system of forces applied in a single point and located in a plane.

The equation of the projections onto the x axis

$$\sum_{k=1}^{n} X_k = T_{BC}\cos 45^\circ - T_{AB}\cos 60^\circ = 0$$

The equation of the projections onto the y axis

$$\sum_{k=1}^{n} Y_k = T_{BC} \sin 45^\circ - T_{AB}$$

Solving this set of equations conjointly and cancelling T_{BC}, we obtain:

$$T_{AB} (\sin 60^\circ + \cos 60^\circ) - T_{BD} = 0$$

whence

$$T_{AB} = \frac{T_{BD}}{\sin 60^\circ + \cos 60^\circ} = \frac{500}{0.866 + 0.5} = 366 \text{ N}$$

Cancelling T_{AB} from the equations, we get:

$$T_{BC} (\cos 45^\circ \sin 60^\circ + \sin 45^\circ \cos 60^\circ) - T_{BD} \cos 60^\circ = 0$$

whence

$$T_{BC} = T_{BD} \frac{\cos 60^\circ}{\sin 105^\circ} = 500 \frac{0.5}{0.966} = 259 \text{ N}$$

Example 6. A bracket consists of a horizontal bar AD of a weight $G_1 = 150$ N which is hingedly fixed to a wall and a brace CB of a weight $G_2 = 120$ N which is also hingedly fixed to the bar AD and wall (Fig. 46*a*). The dimensions of the bracket are indicated in the figure. A weight $G = 300$ N is suspended from the end of bar in point D. Find the reactions of the hinges A and C.

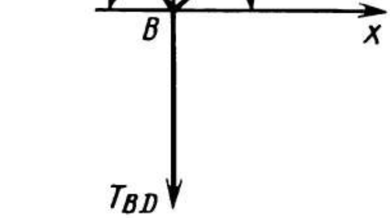

Fig. 45.
To example 5

Solution. Rejecting the external constraints, we consider the equilibrium of the entire bracket. It is acted upon by the given forces G_1, G_2, G and constraint reactions X_A, Y_A, X_C, and Y_C which are unknown. When freed from external constraints, the bracket is not a rigid structure (the bars can turn on the hinge B), but according to the stiffening principle, the forces acting on it in

equilibrium should satisfy the conditions of equilibrium in statics.

Having set up these conditions, we find:

$$\sum X_k = X_A + X_C = 0$$
$$\sum Y_k = Y_A + Y_C - G_1 - G_2 - G = 0$$
$$\sum m_A(F_k) = X_C 4a - Y_C a - G_2 a - G_1 2a - G4a = 0$$

These three equations contain four unknowns: X_A, Y_A, X_C, and Y_C.

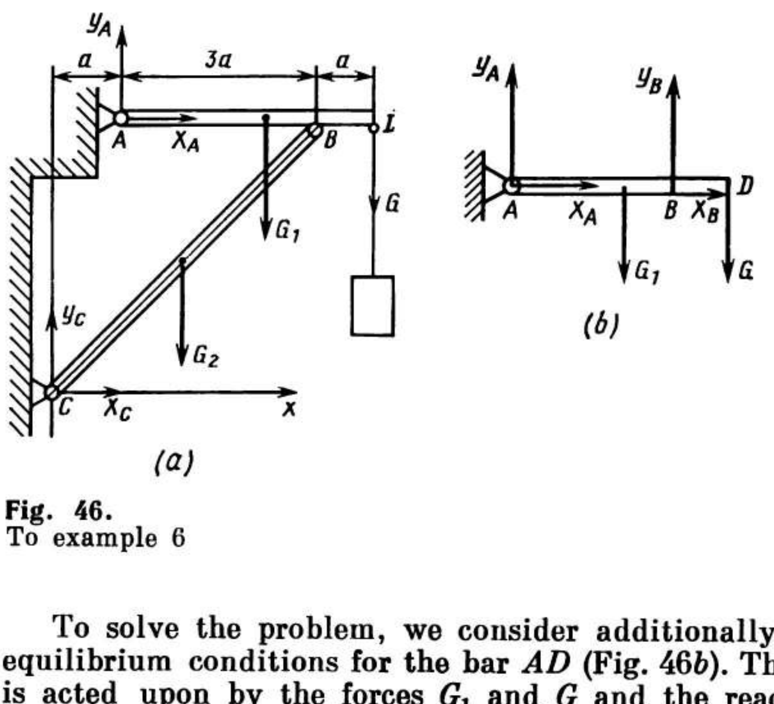

Fig. 46.
To example 6

To solve the problem, we consider additionally the equilibrium conditions for the bar AD (Fig. 46*b*). The bar is acted upon by the forces G_1 and G and the reactions X_A, Y_A, X_B, and Y_B.

The missing fourth equation can be set up by taking the moments of these forces relative to a centre B (in that case, the new unknowns X_B and Y_B will not enter the equation):

$$\sum m_B(G_k) = -Y_A 3a + G_1 a - Ga = 0$$

Solving the system of four equations (beginning with the last) we can then find:

$$Y_A = \frac{1}{3}(G_1 - G) = -50 \text{ N}, \; Y_C = \frac{2}{3}G_1 + G_2 + \frac{4}{3}G = 620 \text{ N},$$

$$X_C = \frac{2}{3}G_1 + \frac{1}{2}G_2 + \frac{4}{3}G = 560 \text{ N}, \; X_A = -X_C = -560 \text{ N}$$

These results suggest that the directions of forces $\mathbf{X}_A$ and $\mathbf{Y}_A$ are opposite to those shown in Fig. 46. The reactions of the hinge B, if these are to be determined, can be found from the equations of the projections of the forces acting on the bar AD onto the axes x and y:

$$X_B = -X_A; \quad Y_B = G_1 + G - Y_A = 500 \text{ N}$$

Example 7. A beam with one end built-in in a wall is loaded by a couple of forces with the moment $m = 10\,000$ N m and a force $F = 5000$ N applied in point C. The length of the beam is $l = 2$ m (Fig. 47). Determine the reactions of the built-in end of beam neglecting the weight of the beam proper.

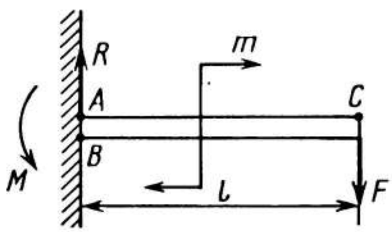

Fig. 47. To example 7

Solution. The beam is acted upon by a couple of forces with a negative moment m and by force F which also produces a negative moment relative to point B in section AB. Therefore, the reactions of the built-in end of beam reduce to a single force R and a positive reaction moment M.

The equations of equilibrium for all forces acting on the beam are set up as follows:

$$\sum X_k = F - R = 0$$

$$\sum m_A(F_k) = -m - Fl + M = 0$$

From the first equation, we find that $R = F = 5000$ N and from the second, that the reaction moment

$$M = m + Fl = 10\,000 + 5000 \times 2 = 20\,000 \text{ N m}$$

Example 8. A couple of forces with moment m_0 is applied to a horizontal beam AB (Fig. 48). Determine the support reactions R_A and R_B and the span between the supports $AB = l$.

Solution. Consider the equilibrium of the beam. The effect of constraints can be replaced by reactions R_A

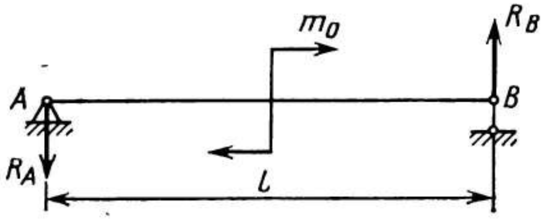

Fig 48.
To example 8

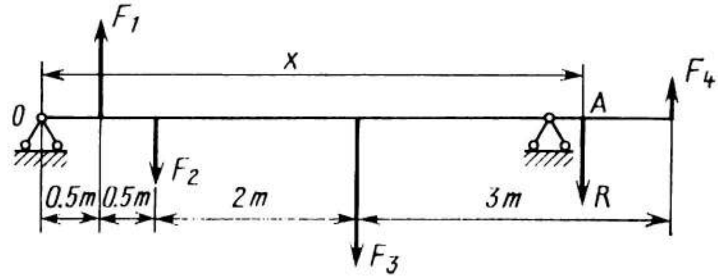

Fig. 49.
To example 9

and R_B. The equation of equilibrium will be written as ollows:

$$\sum m_A = -m_O + R_B l = 0$$

whence

$$R_B = \frac{m_O}{l}$$

A couple of forces can only be balanced by another couple, because of which the reaction R_A is equal to reaction R_B and directed in the opposite sense:

$$R_A = R_B = \frac{m_O}{l}$$

Example 9. A homogeneous beam is under the action of a system of vertical forces $F_1 = 4000$ N, $F_2 = 2000$ N, $F_3 = 4500$ N, and $F_4 = 1000$ N (Fig. 49). The dimensions are indicated in the figure. The weight of the beam proper can be neglected.

Find the distance x from the end O of the beam, at which the resultant of the vertical forces acts on the beam.

Solution. The resultant is found as the algebraic sum of the forces:

$$R = \sum F_k = F_1 - F_2 - F_3 + F_4$$

or numerically

$$R = 4000 - 2000 - 4500 + 1000 = -1500 \text{ N}$$

The resultant is directed downwards, since

$$F_2 + F_3 > F_1 + F_4$$

The sum of the moments of component forces relative to the centre is:

$$Em_0(F_k) = F_1 \times 0.5 - F_2 \times 1 - F_3 \times 3 + F_4 \times 6 = 4000 \times \times 0.5 - 2000 \times 1 - 4500 \times 3 + 1000 \times 6 = -7500 \text{ N m}$$

According to Varignon's theorem,

$$m_O(R) = -RX = \sum m_O(F_k)$$

whence

$$x = -\frac{\sum m_O(F_k)}{R} = -\frac{7500}{-1500} = 5 \text{ m}$$

Example 10. A beam (Fig. 50) of a length $DC = l$ is loaded by two forces $F_1 = 40$ kN and $F_2 = 30$ kN and a couple of forces with the moment $m = 20$ kN m. The weight of the beam is $G = 10$ kN. The beam dimensions are: $DC = 9$ m, $DA = 2$ m, $AB = 6$ m, and $AE = 2$ m. Determine the support reactions.

Solution. We reject the constraints and replace their action by corresponding reactions. The reaction R_B of support B is directed perpendicular to the support plane,

i.e. vertically. The reaction of support A can be resolved into two components: a horizontal and a vertical, i.e. directed along the coordinate axes. The weight of the beam is assumed to be applied in the mid of length DC

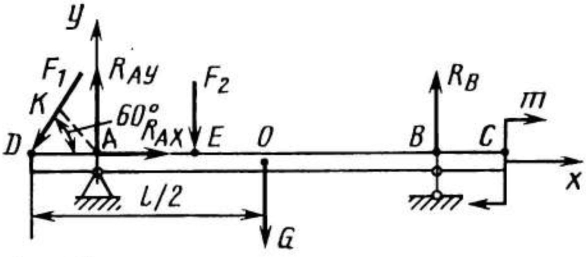

Fig. 50.
To example 10

and the beam is assumed to be homogeneous. The equations of equilibrium of the forces can be set up as follows:

$$\sum m_k(A) = 0; \quad F_1AK - F_2AE - GAO + R_BAB - m = 0$$

$$\sum X_k = 0; \quad -F_1 \cos 60° + R_{AX} = 0$$

$$\sum Y_k = 0; \quad -F_1 \cos 30° + R_{AY} - F_2 - G + R_B = 0$$

where AK is the perpendicular drawn from the centre of moments onto the line of force application:

$$AK = DA \sin 60° = 2 \times 0.866 = 1.73 \text{ m}$$

The reactions are then determined:
from the first equation

$$R_B = \frac{-F_1AK + F_2AE + GAO + m}{AB}$$

$$= \frac{-40 \times 1.73 + 30 \times 2 + 10 \times 2.5 + 20}{6} = 18.68 \text{ kN}$$

from the second equation

$$R_{AX} = F_1 \cos 60° = 40 \times 0.5 = 20 \text{ kN}$$

and from the third equation

$$R_{AY} = F_1 \cos 30° + F_2 + G - R_B$$

$$= 40 \times 0.866 + 30 + 10 - 18.66 = 56 \text{ kN}$$

Example 11. A horizontal shaft AB carries a wheel C of a diameter of 1 m and wheel D of a diameter of 0.2 m (Fig. 51). Other dimensions are indicated in the figure. A vertical force $F_1 = 15$ kN is applied tangentially to the wheel C and a horizontal force F_2 is applied tangen-

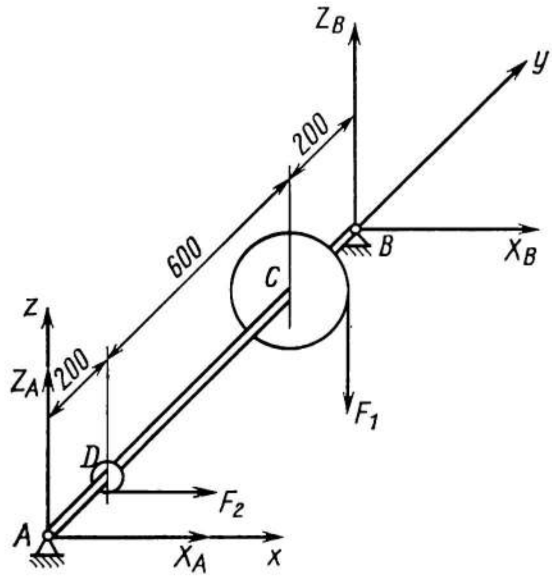

Fig. 51.
To example 11

tially to the wheel D. Find the force F_2 and the reactions of the bearings A and B in equilibrium.

Solution. The origin of coordinates can be placed in point A, with the y axis directed along the shaft axis and the axes x and z, perpendicular to it horizontally and vertically.

The shaft is in equilibrium under the action of the forces F_1 and F_2 and the bearing reactions R_A and R_B. The reactions of the bearings can be resolved into components along the axes x and z: X_A, Z_A, X_B, and Z_B. There are no components along the y axis, since there are no constraints (and therefore, their reactions) which could restrict shaft displacements along that axis.

To find the five unknowns (F_2, X_A, Z_A, X_B, Z_B), we set up the equilibrium equations. Two equations can be obtained by equating to zero the sum of the projections of all forces onto the axes x and z and three equations, by equating to zero the sum of the moments of all forces relative to the axes x, y, and z:

$$\sum X_k = F_2 + X_A + X_B = 0 \tag{a}$$

$$\sum Z_k = -F_1 + Z_A + Z_B = 0 \tag{b}$$

$$\sum m_x(F) = -F_1 \times 0.8 + Z_B \times 1 = 0 \tag{c}$$

$$\sum m_y(F) = -F_2 \times 0.1 + F_1 \times 0.5 = 0 \tag{d}$$

$$\sum m_z(F) = -F_2 \times 0.2 - X_B \times 1 = 0 \tag{e}$$

We find from equation (d) that $F_2 = 5F_1 = 75$ kN, then from equation (e) that $X_B = -0.2F_2 = -15$ kN and from equation (a) that $X_A = -F_2 - X_B = -60$ kN, from equation (c) that $Z_B = 0.8F_1 = 12$ kN and, finally, from equation (b) that $Z_A = F_1 - Z_B = 3$ kN.

The minus sign at X_B and X_A implies that these reaction components are directed in the opposite sense to that indicated in the figure.

2.8. Geometrical Characteristics of Plane Sections

As will be demonstrated later in the book, the strength of structural elements in some kinds of deformation (such as torsion, bending) depends not only on the cross-sectional area of an element, but also on some other geometrical characteristics.

Let us consider the principal geometrical characteristics of plane sections which determine the resistance of structural elements to the action of torsional and bending loads: static moments of inertia, and moments of resistance.

The static moment of the area of a plane section relative to an axis lying in the same plane is the sum of the products of the areas of elementary planes by their dis-

tances from that axis. The static moment of an area will be denoted by S with a subscript of a corresponding axis:

$$S_x = \int\limits_S y\,dS; \qquad S_y = \int\limits_S x\,dS$$

On the basis of the theorem of the moment of a resultant force, the sum of static moments of elementary planes relative to a particular axis is equal to the moment of

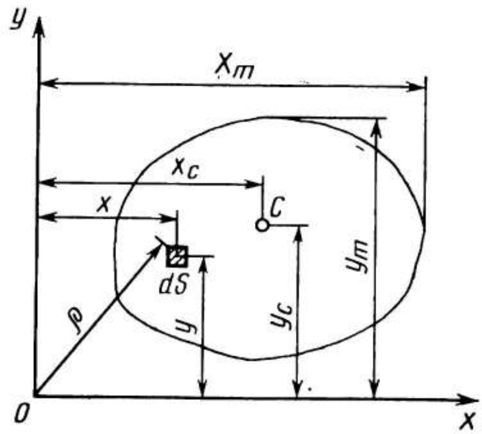

Fig. 52.
Geometrical characteristics of plane sections

the whole area relative to that axis, i.e. to the product of the cross-sectional area by the distance from the centre of gravity of the section to the axis:

$$S_x = \int\limits_S y\,dS = y_c S$$

where y_c is the distance from the centre of gravity C of the section considered to the x axis (Fig. 52).

The static moment relative to the y axis

$$S_y = x_c S$$

where x_c is the distance from the centre of gravity of the section to the y axis.

The static moment has the dimensions of the third power of length, i.e. is measured in mm^3, cm^3, m^3. The

static moment may be either positive or negative, depending on the quadrant of the coordinate system in which the centre of gravity is located.

The axes passing through the centre of gravity of a plane section are called the central axes. The static moments of plane sections relative to the central axes are always equal to zero. Indeed, with $x_c = 0$ and $y_c = 0$ we obtain

$$S_x = S \times 0 = 0 \quad \text{and} \quad S_y = S \times 0 = 0$$

If a section has an axis of symmetry, this axis always passes through the centre of gravity of the figure, and therefore, the static moment of the section relative to the axis of symmetry is always equal to zero. When calculating the static moment of an intricate section, the latter is divided into simple portions and the static moments of these portions are added algebraically:

$$S_x = S_{1x} + S_{2x} + \ldots + S_{nx}$$

where S_x is the static moment of the whole section relative to the x axis; and $S_{1x}, S_{2x}, \ldots, S_{nx}$ are the static moments of simple sections.

The polar moment of inertia of a plane section relative to a certain centre lying in the section plane is the sum of the products of elementary areas by the squares of their distances from that centre, taken for the entire area of the section:

$$J_p = \int_S \rho^2 \, dS$$

where J_p is the polar moment of inertia of the section relative to centre O; and ρ is the distance from an elementary area dS to that centre (see Fig. 52).

The axial moment of inertia of a plane section relative to a certain axis line in the section plane is the sum of the products of elementary areas by the squares of their distances from that axis, taken for the whole area of a figure:

$$J_x = \int_S y^2 \, dS; \qquad J_y = \int_S x^2 \, dS$$

where J_x and J_y are the axial moments of inertia relative to the axes x and y; and x and y are the coordinates of the elementary areas (see Fig. 52).

The moments of inertia relative to the central axes are called the central moments of inertia. As may be seen from Fig. 52, $\rho^2 = x^2 + y^2$, and therefore,

$$J_p = \int_S \rho^2 \, dS = \int_S x^2 \, dS + \int_S y^2 \, dS$$

or

$$J_p = J_x + J_y$$

Thus, the sum of axial moments of inertia relative to two perpendicular axes is equal to the polar moment of intertia relative to the point of intersection of these axes.

The polar and axial moments of inertia of a section are always positive and cannot be equal to zero. They are measured in the units of length to the power of four.

The centrifugal moment of inertia, more properly called the product of inertia, of a plane section relative to two mutually perpendicular axes x and y lying in the section plane is the sum of the products of elementary areas by their coordinates, taken for the whole area of the section:

$$J_{xy} = \int_S xydS$$

The product of inertia may be positive, negative or equal to zero depending on the position of the axes. The two mutually perpendicular axes, relative to which the product of inertia is equal to zero, are called the main axes of inertia.

The axis of symmetry of a figure is the main axis of that figure, since each positive quantity $xydS$ in the first and third quadrants have corresponding negative quantities in the second and fourth quadrants (Fig. 53).

Another main axis is perpendicular to the axis of symmetry. The main axes passing through the centre of

gravity of a section are termed the main central axes of that section. If a section has two axes of symmetry, they are the main central axes. If a section has more than two axes of symmetry (such as a circle), then any two mutually perpendicular axes passing through the centre of gravity of the section are the main central axes.

The polar moment of resistance of a plane section relative to a certain centre lying in the section plane is the ratio of the moment of inertia relative to that centre to the distance from the centre to the most remote point of the section:

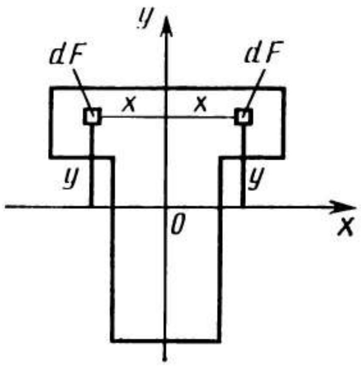

Fig. 53.
Main axes of inertia

$$W_p = \frac{J_p}{\rho_m}$$

where W_p is the polar moment of resistance, and ρ_m is the distance from the centre to the most remote point of the section.

The axial moment of resistance of a plane section relative to a certain axis lying in the section plane is the ratio of the moment of inertia relative to that axis to the distance from the axis to the most remote point of the section:

$$W_x = \frac{J_x}{Y_m}; \qquad W_y = \frac{J_y}{X_m}$$

where W_x and W_y are the moments of resistance of the section relative to the axes x and y, and Y_m and X_m are the distances from these axes to the most remote points of the section.

Moments of resistance are measured in third powers of units of length. Let us show how to calculate the moments of inertia and resistance for simple sections often encountered in calculation practice. The moments of inertia of more intricate sections can be found as sums of the moments of inertia of the simple sections into which an intricate section can be divided.

Rectangle. Find the axial moment of inertia of a rectangle with the base b and height h relative to the axes x and y passing to the centre of gravity O of the rectangle parallel to its sides (Fig. 54). Let us separate an ele-

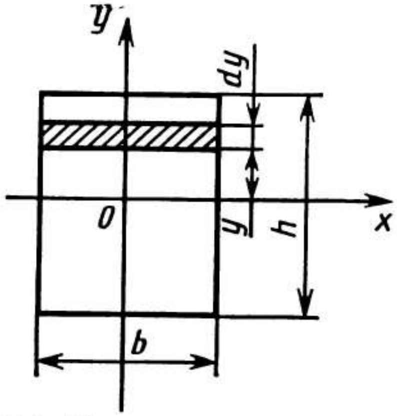

Fig. 54.
Moment of inertia of a rectangle

mentary area with the base b and height dy at a distance y from the x axis. Since $dS = bdy$, then

$$J_x = \int_S y^2 bdy$$

Integrating between $-h/2$ and $h/2$, we get:

$$J_x = \int_{h/2}^{h/2} y^2 bdy = \frac{bh^3}{12}$$

The moment of inertia relative to the y axis is found in a similar way:

$$J_y = \frac{hb^3}{12}$$

Substituting in these formulae $b = h = a$, we obtain the moment of inertia for a square with the side a:

$$J_x = J_y = \frac{a^4}{12}$$

If $Y_{max} = h/2$ and $X_{max} = b/2$, the moments of resistance of a rectangle can be found by the formulae:

$$W_x = \frac{bh^2}{6}; \quad W_y = \frac{hb^2}{6}$$

For a square with the side a, we get:

$$W_x = W_y = \frac{a^3}{6}$$

Circle and annulus. To determine the polar moment of inertia of a circle, let us draw two concentric circumferences with the radii ρ and $\rho + d\rho$ so as to separate an

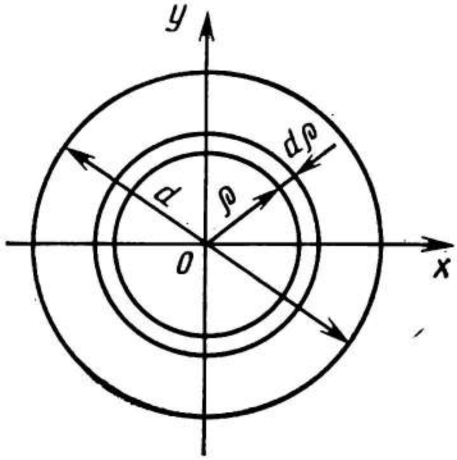

Fig. 55.
Moments of inertia of a circle and annulus

annulus (Fig. 55). The area of this annulus, or ring, is equal to the product of the length of circumference by the width of ring:

$$dS = 2\pi\rho d\rho$$

and the polar moment of inertia:

$$J_p = \int_0^{d/2} 2\pi\rho d\rho \cdot \rho^2 = 2\pi \int_0^{d/2} \rho^3 d\rho$$

Upon integration and substitution of the limits, the polar moment of inertia of a circle relative to its centre

O will be found by the formula:

$$J_p = \frac{\pi d^4}{32} \approx 0.1 d^4$$

To find the axial moments of inertia relative to the axes x and y, note that for a circle $J_x = J_y$; then we have

$$J_x = J_y = \frac{J_p}{2} = \frac{d^4}{64} \approx 0.05 d^4$$

The moments of resistance of a circle can be determined by the formulae:

$$W_p = \frac{J_p}{\frac{d}{2}} = \frac{\pi d^4}{\frac{d}{2} 32} \approx 0.2 d^3; \quad W_x = W_y = \frac{\pi d^3}{32} = 0.1 d^3$$

The polar moment of inertia of a circular ring with the outside diameter D and inside diameter d can be determined by the formula given above, but the integration should be done within the limits of $d/2$ and $D/2$:

$$J_p = 2\pi \int_{d/2}^{D/2} \rho^3 \, d\rho$$

which gives:

$$J_p = \frac{\pi}{32} (D^4 - d^4) \approx 0.1 (D^4 - d^4)$$

The axial moments of inertia of a ring can be found by the formula:

$$J_x = J_y \frac{\pi}{64} (D^4 - d^4) \approx 0.05 (D^4 - d^4)$$

The moments of resistance of a ring will be found by the formulae:

the polar moment

$$W_p = \frac{\pi}{16D} (D^4 - d^4) \approx \frac{0.2}{D} (D^4 - d^4)$$

and the axial moment

$$W_x = W_y = \frac{\pi}{32D} (D^4 - d^4) \approx \frac{0.1}{D} (D^4 - d^4)$$

The geometrical characteristics of simple sections are given in Appendix I.

In some cases it is needed to calculate the moment of inertia of an intricate section relative to an axis which is parallel to the axis for which the moment of inertia is known. Suppose that we have to determine the moment

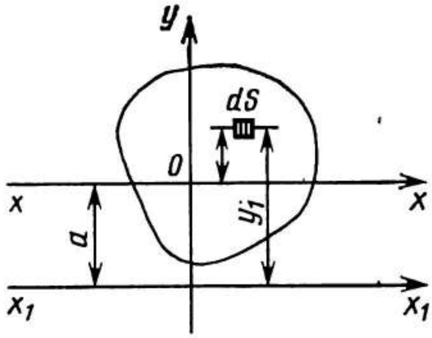

Fig. 56.
Axial moment of inertia with **parallel transfer** of axes

of inertia J_{x1} of a section relative to the axis x_1 which is parallel to the central axis x and spaced from the latter at a distance a (Fig. 56).

The known moment of inertia J_x and the sought-for moment J_{x1} can be written as follows:

$$J_x = \int\limits_S y^2\,dS; \qquad J_{x1} = \int\limits_S y_1^2\,dS$$

As may be seen from Fig. 56, the distance of all elementary areas dS from the new axis x_1 is greater by the value a than their distance from the initial axis, i.e. $y_1 = y + a$; then

$$J_{x1} = \int\limits_S (y+a)^2\,dS = \int\limits_S y^2\,dS + 2a\int\limits_S y\,dS + a^2\int\limits_S dS$$

The first integral in this formula is the central moment of inertia J_x; the second integral is equal to zero, since it is essentially the static moment of the area of the figure relative to the x axis which passes through the

centre of gravity; and the third integral is equal to the product a^2S. Therefore

$$J_{x1} = J_x + a^2S$$

i.e. the moment of inertia of a section relative to an axis is equal to the moment of inertia relative to the parallel axis, which passes through the centre of gravity, plus the product of the area of the section by the square of the distance between the axes.

Les us now derive the formula for the product of inertia on passage to parallel axes. Suppose that the

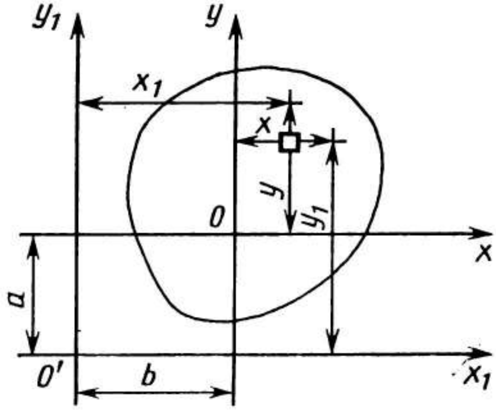

Fig. 57.
Product of inertia with parallel transfer of axes

product of inertia J_{xy} of a figure (Fig. 57) relative to the central axes x and y is known. It is required to determine the product of inertia of the figure, J_{x1y1}, relative to new axes x_1 and y_1 which are parallel to the central axes and spaced from the latter respectively at distances a and b. As may be seen from Fig. 57, these distances are essentially the coordinates of the centre of gravity of the section in the coordinate system x_1, y_1.

The known product of inertia relative to the central axes

$$J_{xy} = \int_S xy\, dS$$

and the sought-for product of inertia relative to the axes x_1 and y_1

$$J_{x1y1} = \int_S x_1 y_1 \, dS$$

The new coordinates of elementary areas can be expressed in terms of the initial coordinates: $x_1 = x + b$ and $y_1 = y + a$.

Substituting them into the formula for J_{x1y1}, we obtain:

$$J_{x1y1} = \int_S (x+b)(y+a) \, dS$$
$$= \int_S xy \, dS + b \int_S y \, dS + a \int_S x \, dS + ab \int_S dS$$

In this formula, the first integral is equal to J_{xy} and the second and third integrals are equal to zero, since they are essentially the static moments relative to the axes passing through the centre of gravity of the section. Thus, the formula can be written in the following form:

$$J_{x1y1} = J_{xy} + abS$$

This formula suggests the definition: the product of inertia of a section relative to arbitrary axes parallel to the central axes is equal to the product of inertia relative to the central axes plus the product of the area of the section by the respective distances between the parallel axes.

2.9. Determination of Centres of Gravity

Consider a system of parallel and likely directed forces $\mathbf{F}_1$, $\mathbf{F}_2$, . . ., $\mathbf{F}_n$ applied to a body (Fig. 58). The resultant of these forces will be directed in the same sense as the component forces and is equal to

$$R = \sum \mathbf{F}_k$$

If each of these forces in the system is turned around the point of application in the same direction and by the same angle, there will be formed new systems of likely directed parallel forces of the same magnitudes and with the same points of application, but having a different general direction. The line of application of the resultant of these systems of parallel forces will always pass through one and the same point.

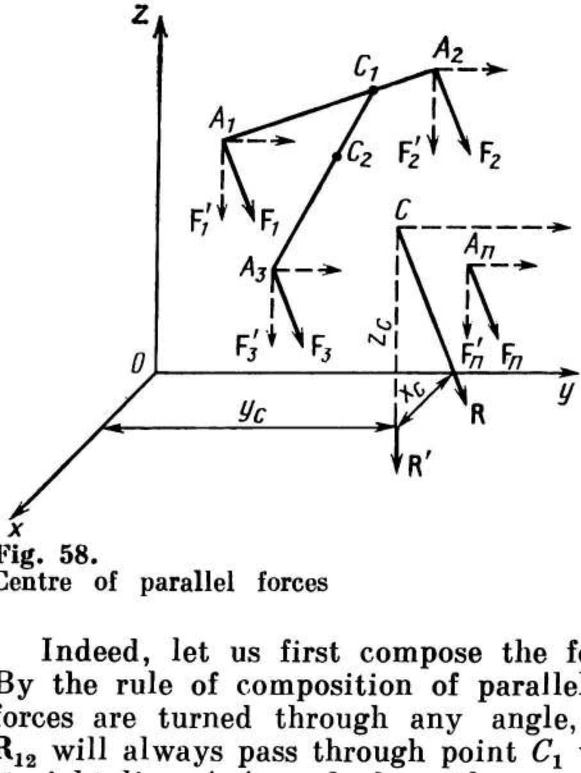

Fig. 58.
Centre of parallel forces

Indeed, let us first compose the forces $\mathbf{F}_1$ and $\mathbf{F}_2$. By the rule of composition of parallel forces, as these forces are turned through any angle, their resultant $\mathbf{R}_{12}$ will always pass through point C_1 which lies on the straight line A_1A_2 and obeys the equality

$$\mathbf{F}_1 A_1C_1 = \mathbf{F}_2 A_2C_1$$

since rotation of the forces can change neither the position of line A_1A_2 nor the equality. By composing the force $\mathbf{R}_{12}$ with force F_3 we find that their resultant will always pass through a similarly determined point C_2 which lies on the straight line C_1A_3, and so on. The composition of all the forces will convince us that their re-

sultant **R** always passes through one and the same point C whose position relative to the points A_1, A_2, . . ., A_n, i.e. to the body, is invariable.

The point C through which the resultant of a system of parallel forces passes always as these forces are rotated around their points of application in the same sense and through the same angle is called the centre of parallel forces.

In order to find the coordinates of the centre of parallel forces, let us take arbitrary coordinate axes x, y, z and denote the coordinates of a point C in this system as x_c, y_c, and z_c. Noting that the position of point C is independent of the direction of the forces, we can turn these forces around their points of application so as to arrange them parallel to the axis Oz and apply the Varignon theorem to them.

Since $\mathbf{R}'$ is the resultant of these forces, we can write their moments relative to the axis Oy:

$$m_y(\mathbf{R}') = \sum m_y(\mathbf{F}'_k)$$

As may be seen from Fig. 58, $m_y(\mathbf{R}') = Rx_c$, since $R' = R$. Similarly, $m_y\mathbf{F}'_1 = \mathbf{F}_1x_1$, since $\mathbf{F}'_1 = \mathbf{F}_1$, etc. Therefore,

$$Rx_c = F_1x_1 + F_2x_2 + \ldots + F_nx_n$$

whence the coordinate of the centre of parallel forces is found:

$$x_c = \frac{F_1x_1 + F_2x_2 + \ldots + F_nx_n}{R} = \frac{\sum F_kx_k}{R}$$

Having written the moments relative to the axis Ox, we obtain a similar formula for the coordinate y_c. In order to determine the coordinate z_c, let us turn all the forces parallel to the axis Oy and write the equations of moments relative to the axis Ox:

$$-Rz_c = -F_1z_1 + (-F_2z_2) + \ldots + (-F_nz_n)$$

whence, it is possible to find z_c.

Thus, we have obtained the following formulae for the centre of parallel forces:

$$x_c = \frac{\sum F_k x_k}{R}; \quad y_c = \frac{\sum F_k y_k}{R}; \quad z_c = \frac{\sum F_k z_k}{R}$$

where R is determined by the equation $R = \sum F_k$.

Any particle of a body near the Earth's surface is acted upon by a downward vertical force which is called the force of gravity.

The resultant of gravity forces of all particles of a body is called the force of gravity of the body.

With any changes in the position of a body in space, the gravity forces of its particles will be always applied in the same points and will be parallel to one another. Their resultant, i.e. the force of gravity, G, of a body, always passes through one and the same point C, the centre of parallel gravity forces of the body which is called the centre of gravity.

The centre of gravity of a body is a fixed point of that body through which the resultant of the gravity forces of all particles of the body passes in any position of the body in space.

The coordinates of the centre of gravity can be found, as for the centre of parallel forces, by the following formulae:

$$x_c = \frac{\sum G_k x_k}{G}; \quad y_c = \frac{\sum G_k y_k}{G}; \quad z_c = \frac{\sum G_k z_k}{G}$$

where x_k, y_k, z_k are the coordinates of the points of application of gravity forces of the body, and G_k is the gravity force of each elementary particle of the body.

It should be emphasized that the centre of gravity of a body is merely a geometrical point and may lie even beyond the contours of a body. For a homogeneous body, the weight G_k of any of its portions is proportional to the volume V_k of that portion: $G_k = \gamma V_k$, and the weight G of the whole body is proportional to the volume V of the body, i.e. $G = \gamma V$, where γ is the weight of a unit of volume, which is called the specific gravity.

Substituting for G and G_k in the formulae above, we obtain:

$$x_c = \frac{\sum V_k x_k}{V}; \quad y_c = \frac{\sum V_k y_k}{V}; \quad z_c = \frac{\sum V_k z_k}{V}$$

Hence, it follows that the coordinates of the centre of gravity of a homogeneous body do not depend on specific gravity γ, and therefore, depend only on the volume occupied by the body and on the body's shape. For that reason, the centre of gravity of a homogeneous body is called the centre of gravity of the volume.

By similar reasoning, it can be easily found that the coordinates of the centre of gravity for a plane section are:

$$x_c = \frac{\sum S_k x_k}{S}; \quad y_c = \frac{\sum S_k y_k}{S}$$

where S is the area of the entire section, and S_k is the area of a kth portion of that section.

If a homogeneous body has the same cross-sectional area along its length and if its cross-sectional dimensions are small compared to the length, the body can be regarded as a material line. The formula for determining the coordinates of the centre of gravity of a line is:

$$x_c = \frac{\sum l_k x_k}{L}$$

where L is the total length of a line, and l_k is the length of its kth element.

If a homogeneous body possesses a plane, axis of centre of symmetry, its centre of gravity lies respectively in the plane, axis or centre of symmetry. This theorem has the following consequences:

(a) the centre of gravity of a parallelogram lies in the point of intersection of the diagonals;

(b) the centre of gravity of a line section lies in the middle of the line; and

(c) the centres of gravity of a regular polygon, circle, ellipse, and sphere lie in their geometrical centres.

The determination of the centres of gravity for some forms of homogeneous bodies will be demonstrated below.

1. **Arc of circle.** Consider a circular arc AB of radius R and central angle $AOB = 2\alpha$ (Fig. 59). Let us place the origin of coordinates in the centre O of the circle and direct the x axis along the bisector of the central angle. Since this axis is the axis of symmetry of the arc, the centre of gravity lies in a certain point C of that axis, i.e. its position is completely determined by a single coordinate x_c. Let us divide the arc AB into elementary sections. The coordinate of the centre of gravity of a homogeneous line is determined by the formula:

$$x_c = \frac{\sum l_k x_k}{L}$$

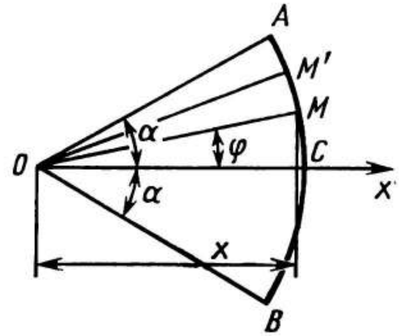

Fig. 59.
Centre of gravity of an arc

Let us increase infinitely the number of elements for division of arc AB. In the limit, they will degenerate to points. Then the numerator of the formula will change to a definite integral taken for the entire length of the arc:

$$x_c = \frac{\lim \sum l_k x_k}{l} = \frac{1}{l} \int_{AB} x \, dl$$

As an element MM' of arc AB is diminished, the points M and M' may be considered to be very close to each other and, therefore, the position of the element MM' is determined by the angle φ, the coordinate of its centre of gravity is $x = R \cos \varphi$, and the length of the element is $dl = Rd\varphi$. Substituting for x and dl in the integrand, we obtain:

$$\int_{AB} x \, dl = \int_{-\alpha}^{+\alpha} R \cos \varphi R \, d\varphi = 2R^2 \sin \alpha$$

Noting that the length of arc is $l = 2\alpha R$ we finally get

$$x_c = \frac{1}{l} \int_{AB} x\, dl = \frac{2R^2 \sin \alpha}{2R\alpha} = \frac{R \sin \alpha}{\alpha}$$

where α is measured in radians.

2. **Triangle.** Let us divide a triangle ABD by straight lines parallel to the side AD into n narrow strips (Fig. 60). It is clear that the centres of gravity of these strips lie on the median BE of the triangle. Therefore, the centre of gravity of the triangle also lies on the median. A similar result can be obtained for all other medians. Thus, it can be concluded that the centre of gravity of a plane triangle lies in the point of intersection of its medians and the intersection point divides each median in a proportion 1:3 (for instance, $CE = \frac{1}{3} BE$).

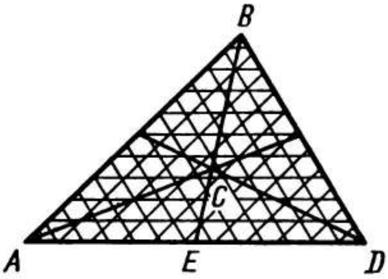

Fig. 60.
Centre of gravity of a triangle

3. **Area of a circular sector.** Consider a circular sector OAB with the radius R and central angle 2α (Fig. 61). Let us divide the area of the sector by radii drawn from the centre O into n equal portions (elementary sections). With an infinite increase of the number n, these sectors can in the limit be regarded as plane triangles whose centres of gravity lie on the arc DE of the radius $\frac{2}{3} R$. Therefore, the centre of gravity of sector OAB coincides with the centre of gravity of arc DE whose position can be found by the formula given above. It can be found finally that the centre of gravity of a circular sector lies on its axis of symmetry at a distance from the centre which is equal to

$$x_c = \frac{2}{3} R \frac{\sin \alpha}{\alpha}$$

Formulae for determining the coordinates of centres of gravity of other geometrical bodies can be found in reference books.

In order to determine the position of the centre of gravity of an intricate section, the latter is divided conditionally into a number of simple sections so that the

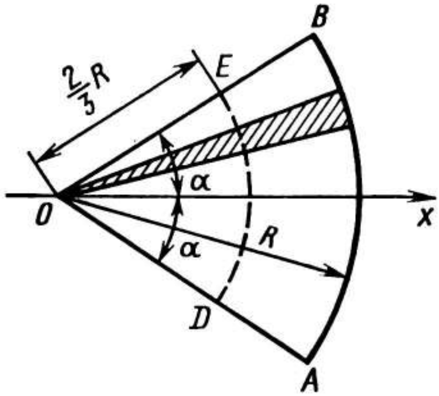

Fig. 61.
Centre of gravity of a circular sector

positions of their centres of gravity could be determined easily. After that the coordinates of the centre of gravity of the entire section are determined.

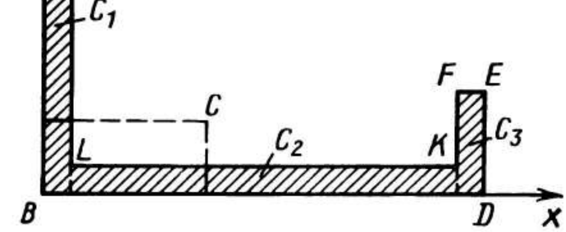

Fig. 62.
Centre of gravity of an intricate-shape section

Example. Determine the coordinates of the centre of gravity of a homogeneous plate shown in Fig. 62. The dimensions of the plate are as follows: $AB = 30$ cm,

$BD = 40$ cm, $ED = 10$ cm, $AN = 2$ cm, $LK = 36$ cm, and $FK = 8$ cm.

Solution. The plate is first divided into three rectangles with their centres of gravity being C_1, C_2, and C_3.

The common centre of gravity can be determined by the formulae

$$x_c = \frac{S_1x_1 + S_2x_2 + S_3x_3}{S}$$

$$y_c = \frac{S_1y_1 + S_2y_2 + S_3y_3}{S}$$

where S_1, S_2, S_3 are the areas of the rectangles; x_i, y_i ($i = 1, 2$, or 3) are the coordinates of the centres of gravity of these rectangles, and S is the total area of the figure.

The results of calculation are given in Table 1, where

Table 1

No. of element	S_i, cm^2	x_i, cm	y_i, cm	$S_{iy} = S_ix_i$, cm^3	$S_{ix} = S_iy_i$, cm^3
1	60	1	15	60	900
2	72	20	1	1440	72
3	20	39	5	780	100
$\sum$	152	—	—	2280	1072

S_{iy} and S_{ix} are the static moments relative to the coordinate axes.

The area and static moments of the entire figure are found by summation:

$S = 152$ cm^2, $S_y = 2280$ cm^3, and $S_x = 1072$ cm^3

The coordinates of the centre of gravity of the plate

$$x_c = \frac{S_y}{S} = \frac{2280}{152} = 15 \text{ cm}$$

$$y_c = \frac{S_x}{S} = \frac{1072}{152} = 7 \text{ cm}$$

The centre of gravity C of the plate can then be constructed by these coordinates.

In cases when a body cannot be divided into portions with known positions of their centres of gravity, the coordinates of its centre of gravity are found by the method of integral calculus.

The centres of gravity of inhomogeneous bodies of intricate shape can be determined by experimental methods. One of these methods is based on suspending a body in various points on threads or cables. The direction of a thread or cable from which the body is suspended gives every time the direction of gravity force. The point of intersection of these lines is the centre of gravity of the body.

Review Questions

1. What is the difference between a free and a non-free body?
2. What force is termed the resultant?
3. If a force of a known magnitude and known direction is to be resolved into two components, is it sufficient for the solution of the problem if one of the components is specified in magnitude and the other, in direction?
4. What are in essence the sections which connect the initial point of a force polygon with its corners?
5. What is called the closing side of a force polygon? If the closing side is equal to zero, what does it mean?
6. What is the moment of a force relative to a point? Will the moment of a force change if the force is transferred along the line of its application?
7. When can forces of different magnitude produce equal moments relative to one and the same centre?
8. What is the moment of a couple of forces? How the moment of a couple is measured? What couples are termed equivalent?
9. How to find a couple of forces that will balance a given plane system of couples?
10. When has the moment of a force, relative to a given axis, the greatest numerical value? When is the moment, relative to an axis, equal to zero?
11. How to determine the static moment of a figure in terms of its area and the coordinates of the centre of gravity?

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12. What are the formulae for determining the coordinates of centre of gravity of a figure?

13. Define the axial and polar moments of inertia and the product of inertia. What are the units for their measurement?

14. What are the formulae for the axial central moments of inertia of a circle and circular ring?

15. What axes are called the main central axes of inertia?

16. How can the moment of inertia of an intricate figure be determined?

17. How can the centres of gravity of simple geometrical figures (triangle, square, rectangle, circle, and sector) be determined.

18. How can the centre of gravity of an unsymmetrical figure be determined?

Chapter Three

The Concepts and Definitions of Strength of Materials

3.1. Definition of Internal Forces

External forces produce strains and internal elastic forces in a body, which increase as the external forces are increased.

For strength calculations of structural elements, it is essential to determine the internal forces probable to appear at specified loads. The magnitude of internal forces appearing due to deformation under the action of external forces can be determined by the method of sections. The essence of the method consists in the following four procedures:

(1) the body being analysed is divided mentally by a plane perpendicular to its axis in the point where the internal forces must be determined;

(2) one of the portions of the body is rejected;

(3) the effect of the rejected part is replaced by internal forces so that the remaining part is in equilibrium; and

(4) equilibrium equations are set up for the forces acting on the remaining part of the body and the internal forces are calculated.

Consider the application of the method of sections taking as an example a bar which is held in equilibrium by a system of forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, $\mathbf{F}_4$, and $\mathbf{F}_5$ (Fig. 63).

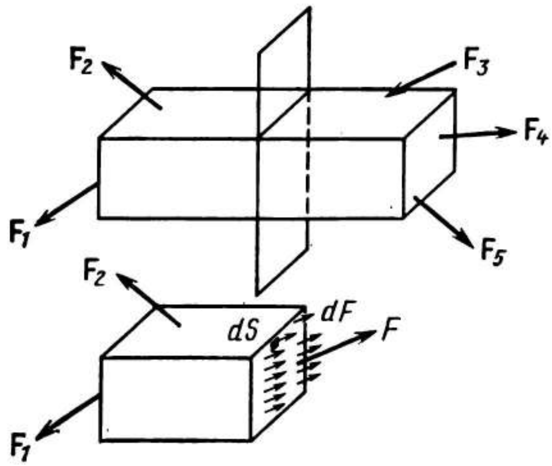


Fig. 63.
Method of sections

Let us cut the body by a plane and reject the right-hand part, replacing the effect of the rejected part on the remaining left-hand part by internal elastic forces. Since these internal forces counter-balance the external forces applied to the cut-off portion of the body, this portion can be calculated for the conditions of static equilibrium. In the general case, these conditions give us six equations of equilibrium:

$$\sum F_x = 0, \quad \sum F_y = 0, \quad \sum F_z = 0,$$
$$\sum m_x(F_k) = 0, \quad \sum m_y(F_k) = 0, \quad \sum m_z(F_k) = 0$$

These equations enable us to determine the static equivalent of the system of internal forces, i.e. to find

the simplest system of forces which can replace the effect of the unknown elastic forces on the section considered. A static equivalent of internal forces can contain various combinations of force factors, including forces and moments, or couples of forces.

In the general case, solution of the equilibrium equations can give three component forces N, Q_z, and Q_y, i.e. the components of the main vector of internal forces directed along the coordinate axes, and three component moments M_x, M_y and M_z, i.e. the components of the main moment of internal forces (Fig. 64), which act in the section considered. The components of the main vector and main moment of internal forces appearing in a cross section of a bar are called the internal force factors acting in that section. The six internal force factors mentioned above are called respectively: N — the longitudinal force; Q_z and Q_y — the lateral forces; M_z and M_y — the bending moments; and M_x — the torsional moment. In particular cases, some of the force factors may be equal to zero.

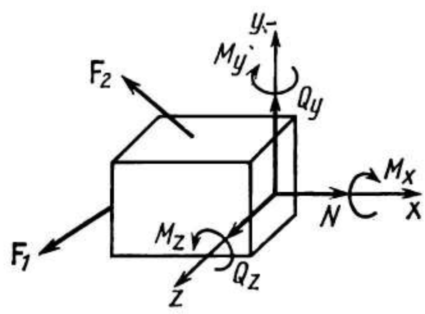


Fig. 64.
Internal force factors

3.2. Stresses

Determination of internal forces does not solve the problem of strength of the body considered, since its strength depends not only on the material and the magnitude of the resultant of internal forces, but also on the pattern of their distribution and the dimensions of the body, in particular, on the dimensions of the section to which the internal forces are applied.

In order to determine the pattern of distribution of internal forces, let us introduce the concept of stress.

Let us divide the cross-sectional area of the body into a large number of small areas and consider one of them, ΔS (Fig. 65). With the dimensions of the area however small, it is acted upon by internal forces distributed all over the section.

Let the resultant ΔF of the internal forces acting on an area considered be directed arbitrarily relative to the area. Taking the ratio of the resultant force to the area of its action we obtain the average stress in that area:

$$p_{av} = \frac{\Delta F}{\Delta S}$$

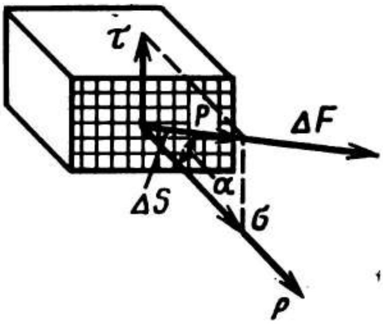


Fig. 65.
Stresses

This expression characterizes the intensity of internal forces in the area selected. For more accurate determination of the intensity of internal forces in a point of a section, the selected area should be as small as possible. In the limit, as ΔS tends to zero, we obtain the actual stress in the given point of a section:

$$p_{\Delta F \to 0} = \lim \frac{\Delta F}{\Delta S} = \frac{dF}{dS}$$

If the internal forces are distributed uniformly over the section, the stress will be determined as the ratio of the resultant of internal forces to the total cross-sectional area:

$$p = \frac{F}{S}$$

Thus, the stress is an internal force that acts per unit area of a section. The dimension of stress is a unit of force divided by the square of a unit of length. The SI unit of stress is a pascal (Pa $=$ N/m^2).

A stress of 1 Pa is very small, because of which a multiple unit, megapascal (MPa) is more often employed. 1 MPa $=$ 1 N/mm^2 $= 10^6$ Pa $\approx$ 10 kgf/cm^2,

In calculations, another multiple unit of stress is sometimes used: a meganewton per square metre ($1\ MN/m^2 = 10^6 N/m^2$). The numerical values of stresses measured in MN/m^2 and N/mm^2 are the same.

The stress in a given point of a section considered is a vector quantity, i.e. it is characterized by magnitude and direction. In the general case, the stress acting in an area dS makes a certain angle α with the normal n to that area.

As is known, the number of differently oriented sections that can be drawn through a given point is infinite, and therefore, it is meaningless to speak about the stress in the given point without specifying the area (section) on which this stress appears. Let us resolve the stress vector into two components: one directed along the normal to the section and the other lying in the section plane (see Fig. 65). The stress component directed along the normal to the section will be called the normal stress and designated by the Greek letter σ. The component lying in the section plane will be called the tangential stress and designated by the Greek letter τ. The stress p can be called the combined stress in the given point of a section.

Considering the stress parallelogram, we obtain:

$$\sigma = p \cos \alpha, \qquad \tau = p \sin \alpha$$

According to the Pythagorean theorem, it can be written that

$$p = \sqrt{\sigma^2 + \tau^2}$$

This resolution of the combined stress has a definite physical meaning. Indeed, a normal stress appears when the loads applied to the body act on particles of its material so as to move them either farther from one another or closer to one another in the direction of the normal to the section, i.e. it appears in tension or compression. Tangential stresses are associated with shear of particles of the material in the plane of the section considered, because of which tangential stresses are alternatively called shear stresses.

In some cases, it is convenient to resolve the vector p into three components directed parallel to the coordinate axes. This resolution for a point in the cross section of a bar is shown in Fig. 66. For these components, the following rule of subscripts has been adopted: the first subsript indicates the coordinate axis to which the normal to the plane of action of the stress considered is parallel; the second subscript gives the coordinate axis to which the given stress is parallel. According to this rule, normal stresses should be written with two subscripts, i.e. σ_{xx}, but usually one or both subscripts are omitted. Similarly, a tangential stress appearing in a cross section can be denoted simply as τ (without subscripts), if its direction is inessential.

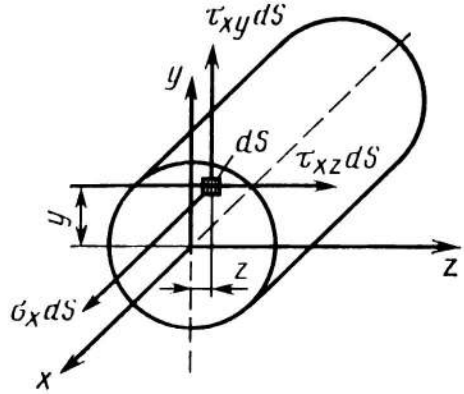


Fig. 66.
Resolution of the total stress into components parallel to the coordinate axes

Let us establish how the stresses and internal force factors in a cross section of a bar are interrelated. Multiplying the stresses σ_x, τ_{xz}, and τ_{xy} by the area dS, we obtain elementary internal forces:

$$dN = \sigma_x dS$$
$$dQ_z = \tau_{xz} dS$$
$$dQ_y = \tau_{xy} dS$$

Summing these elementary forces over the entire area of the section, we find expressions for the components of the main vector of internal forces:

$$N = \int_S \sigma_x \, dS$$

$$Q_z = \int_S \tau_{xz}\, dS$$

$$Q_y = \int_S \tau_{xy}\, dS$$

Multiplying each of the elementary forces by the distance to the corresponding axis, we obtain the moments of internal forces:

$$dM_z = (\sigma_x dS)\, y$$
$$dM_y = (\sigma_x dS)\, z$$
$$dM_x = (\tau_{xz} dS)\, y - (\tau_{xy} dS)\, z$$

Summing the elementary moments over the entire area of the section, we get the expressions for the components of the main moment of internal forces:

$$M_z = \int_S \sigma_x y\, dS$$

$$M_y = \int_S \sigma_x z\, dS$$

$$M_x = \int_S (\tau_{xz} y - \tau_{xy} z)\, dS$$

These formulae will be used in the further discussion for solving one of the principal problems of strength of materials: the determination of stresses by the known internal force factors.

3.3. Strains

Let us separate in a bar an infinitely small element of length dx which is shown in two projections in Fig. 67. Let two equal and oppositely directed forces F be now applied to the ends of the bar exactly along the bar axis. Under the action of these forces the bar will be in equilibrium, but its dimensions will change, i.e. the bar will be deformed. The deformed state of the separated ele-

ment is shown by dotted lines in the figure: the length of the element has increased, whereas the dimensions of its cross sections decreased. Let us denote by $\Delta\, dx$ the increment of the length of the element; this increment is called the absolute elongation. The ratio of the length

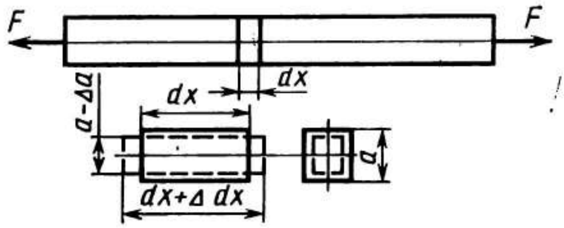


Fig. 67.
Strains

increment of the element to its original length is called longitudinal deformation (strain):

$$\varepsilon = \frac{\Delta\, dx}{dx}$$

Absolute elongation is measured in linear units, like the length of a bar, mm, cm, or m. Relative elongation is dimensionless and is often given as percentage of the original length:

$$\varepsilon\% = \frac{\Delta l}{l}\, 100 = \varepsilon \times 100\%$$

where Δl is the length increment of the bar, and l is its original length.

Longitudinal strain is considered positive in extension and negative in compression.

The ratio of the changed lateral size Δa to its original value is called lateral strain:

$$\varepsilon' = \frac{\Delta a}{a}$$

In tension, the lateral dimensions of a bar diminish and, by the adopted rule of signs, ε' is negative. Longi tudinal and lateral strains are generally called linear strains (or deformations).

The tensile (or compressive) strain considered above can appear in a bar when the internal forces in a cross section are reduced to a resultant force N (Fig. 68) which is equal to the sum of the projections of external forces, located at one side of the section, onto the x axis. If the internal forces acting in the section are reduced merely to

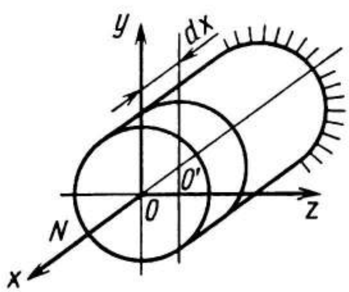


Fig. 68.
Tensile (compressive) strain

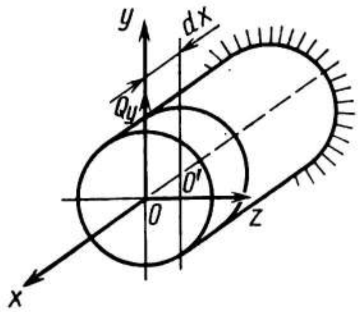


Fig. 69.
Shear strain

a resultant Q_y located in the section plane (Fig. 69), a shear strain will appear.

If all internal forces could be reduced to a resultant Q_z, we had a shear strain as well, but directed along the z axis. In torsional deformation, the internal forces acting in a section are reduced merely to a moment M_x relative to the x axis, which is equal to the sum of the moments of all external forces located at one side of the section (Fig. 70).

The moment M_x is called the torsional moment. In torsion, the axis of a bar remains straight, but the generatrix changes to a helical line.

In bending, the static equivalent of internal forces is the moment M_z relative to the z axis (Fig. 71). In that case, a bar experiences a flexural strain in the vertical plane. In bending, the axis of a bar becomes curved and the deformation is a combination of tension and compression, since each element of the bar is compressed in the lower portion and tensioned in the upper. If the internal forces could be reduced to a moment M_y relative to the y axis, we would have bending in the horizontal plane,

Thus, the nature of strain is completely determined by the internal force factors appearing in a section of a bar. Complex deformations are possible, but any deformation

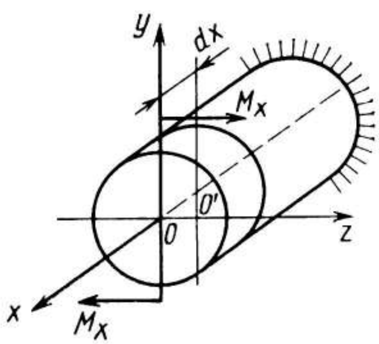


Fig. 70.
Torsional strain

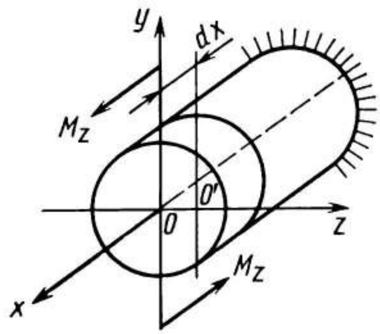


Fig. 71.
Flexural strain

can always be represented as consisting of a number of simple component deformations.

3.4. Stress-strain Relationships

As has been established by the results of experimental tests of specimens made from various structural materials, there exists a direct proportionality between normal stresses acting in cross sections and the strain ε, though to certain limits (Fig. 72). This relationship, which is the principal one in strength of materials, is called Hooke's law and written as

$$\sigma = E\varepsilon$$

Hooke's law can be formulated as follows: the normal stress in tension or compression is directly proportional to the longitudinal strain of a bar.

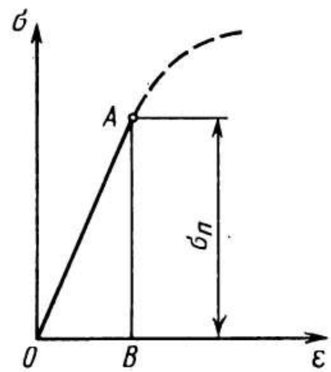


Fig. 72.
Stress-strain diagram in tension (compression)

The law was discovered in 1660 by Robert Hooke, an English physicist.

The proportionality factor E which correlates the normal stress and longitudinal strain is called the modulus of elasticity of a material in tension-compression. It is also called the elastic modulus, modulus of elongation, and Young's modulus. The modulus of elasticity E is one of the principal physical constants which characterize the capability of a material to resist elastic deformation. With a higher value of E, a material will elongate or contract less on the application of the same force F.

Since the strain ε is dimensionless, the elastic modulus E is measured in the same units as the stress σ, i.e. in pascals or megapascals.

For some materials, the modulus of elasticity is the same in tension and compression, whereas for others it is different. In practical calculations of structural elements this difference is usually neglected, i.e. the same value of E is taken for most structural materials both in tension and compression.

Average values of elastic modulus E for selected materials are given in Table 2.

Table 2

Material	Modulus of elasticity E, MPa	Coefficient of lateral strain μ
Steel	$(1.9\text{-}2.15)\times10^5$	0.26-0.33
Cast iron	$(0.8\text{-}1.50)\times10^5$	0.23-0.27
Copper, brass, bronze	$(0.9\text{-}1.30)\times10^5$	0.31-0.34
Aluminium alloys	$(0.69\text{-}0.75)\times10^5$	0.33-0.36
Lead	0.17×10^5	0.42
Glass	0.56×10^5	0.25
Concrete	$(1\text{-}3)\times10^4$	0.16-0.18
Brickwork	$(0.3\text{-}0.5)\times10^4$	0.16-0.34
Wood (along fibres)	$(8.8\text{-}15.7)\times10^3$	—
Fabric-based laminate	$(6\text{-}10)\times10^3$	0.17-0.25
Rubber	(8.0)	0.47
Kapron	$(1.4\text{-}1.9)\times10^3$	—

Experience shows that the linear relationship between stress and strain is only observed to a certain limiting

stress beyond which the σ-ε relationship ceases to be linear. The highest stress at which still there is a proportionality between σ and ε is called the limit of proportionality and denoted σ_{pr}. Thus, Hooke's law is valid only up to the limit of proportionality.

Some materials have a very low proportionality limit and exhibit substantial deviations from Hooke's law already at a low stress. Among them are cast iron, stone, and polymers. On the other hand, steels belong to structural materials having a high proportionality limit.

3.5. Coefficient of Lateral Strain

As has been shown by experiments, the ratio of lateral strain ε' to longitudinal strain ε in tension or compression of a particular material, within the limits of applicability of Hooke's law, is constant. The absolute magnitude of this ratio is called the coefficient of lateral strain, or Poisson's ratio:

$$\mu = \left| \frac{\varepsilon'}{\varepsilon} \right|$$

This ratio was introduced into strength of materials and the theory of elasticity in the 1830's by Poisson of France. Like the modulus of elasticity, Poisson's ratio is a characteristic of elastic properties of a material. For isotropic materials, the elastic modulus and Poisson's ratio are constant for any direction of tensile and compressive forces. For anisotropic materials, i.e. those whose properties are different in different directions, a number of values of these characteristics are taken depending on directions of forces. Such materials include wood, laminated plastics, stone, and fabrics.

Hooke's law is represented graphically in Fig. 72 where stresses are laid off as ordinates and strains as abscissae. The relationship $\sigma = E\varepsilon$ is described by straight line OA.

The coefficients of lateral strain for selected materials deformed within the elastic limit are given in Table 2.

Let us analyse the relationship between the coefficient of lateral strain μ and the volume changes of a bar

in tension or compression. Consider the tension of a bar of length l having a square cross section with the side a.

The initial volume of the bar before tension

$$V = a^2 l$$

Upon tension, the length of the bar increases to $l_1 = l + \Delta l = l + \varepsilon l = l(1 + \varepsilon)$.

The dimensions of the bar cross section have diminished and the square side changes to

$$a_1 = a - \Delta a = a - \varepsilon' a = a(1 - \varepsilon') = a(1 - \mu\varepsilon)$$

Thus, the volume of the bar upon tension is:

$$V_1 = a_1^2 l_1 = a^2(1 - \mu\varepsilon)^2 l(1 + \varepsilon)$$
$$= a^2 l(1 + \varepsilon - 2\mu\varepsilon - 2\mu\varepsilon^2 + \mu^2\varepsilon^2 + \mu^2\varepsilon^3)$$

Rejecting the terms with ε^2 and ε^3 as higher-order infinitesimals, we obtain:

$$V_1 = a^2 l(1 + \varepsilon - 2\mu\varepsilon)$$

The absolute change of the volume

$$\Delta V = V_1 - V = a^2 l\varepsilon(1 - 2\mu)$$

and the relative change of volume

$$\frac{\Delta V}{V} = \frac{a^2 l\varepsilon(1 - 2\mu)}{a^2 l} = \varepsilon(1 - 2\mu)$$

It is clear that the volume of a bar cannot diminish on tension, and therefore, the volume change is positive and $1 - 2\mu \geqslant 0$, whence $\mu \leqslant 0.5$, which has been confirmed by experimental tests of various materials. If the volume of a material is not changed on tension, Poisson's ratio of the material is equal to 0.5.

Review Questions

1. What is stress and what are the units for its measurement?
2. What stresses are called normal and tangential?
3. What is the strain of a body?
4. What is the difference between deformation and displacement?

5. What principal kinds of strain are produced by external forces?

6. What does Hooke's law express?

7. What are the limits within which Hooke's law is applicable?

8. What characterizes the modulus of elasticity? What are the units for its measurement?

9. What is Poisson's ratio and what does it characterize?

Chapter Four

Simple Kinds of Loading

This chapter will deal with simple kinds of loading which produce only one type of internal force factor in the cross sections of a bar: either a force in tension or shear, or a moment in torsion or pure bending.

4.1. Tension and Compression

Tension and compression are the kinds of simple loading in which only one kind of internal force factor, a longitudinal force N, appears in the cross section of a bar.

Let us consider a bar of constant cross-sectional area with two oppositely directed forces applied to its ends

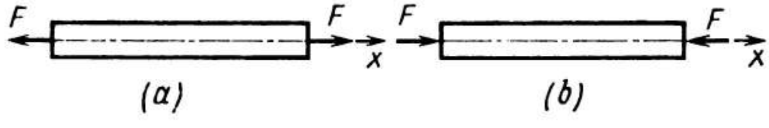


Fig. 73.
Diagram of external forces acting in (a) tension and (b) compression

(Fig. 73). Both in tension (Fig. 73a) and compression (Fig. 73b), the bar is in equilibrium, i.e. the sum of the projections of all forces onto the x axis is equal to zero:

in tension: $-F + F = 0$
in compression: $F - F = 0$

The internal force factor, i.e. the longitudinal force N, can be found by the method of sections: let the bar be cut in the point where the force N is to be determined and the effect of the rejected portion of the bar onto the remaining portion be replaced by the internal force N (Fig. 74).

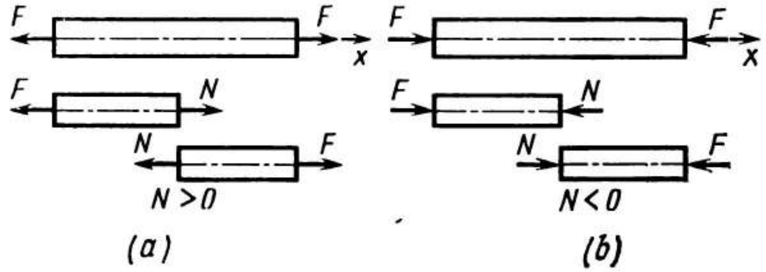


Fig. 74.
Determination of internal forces by the method of sections in (*a*) tension and (*b*) compression

From the condition of equilibrium for any of the portions of the bar it is possible to find the longitudinal force

$$-F + N = 0$$

In tension, the force N (Fig. 74*a*) is directed away from the section and has the plus sign and in compression (Fig. 74*b*), it is directed into the section and has the minus sign.

The pattern of distribution of the longitudinal force along the bar can be represented graphically (Fig. 75). Such a graph is called the loading diagram in strength of materials. In our case, $N = F =$ constant and the graph is essentially a straight line with abrupt changes in the points of application of forces F, i.e. at the ends of the bar, by the magnitude of force (see Fig. 75, *a* and *b*). The sign of a longitudinal force depends on how the bar length changes under the action of the force applied. In tension by forces F (which are positive), a bar of a constant cross-sectional area and of length l is elongated by

Δl (Δl is called the elongation of a bar) and in compression by forces F (which are then negative), it is contracted by Δl (Fig. 76). Dotted lines in the figure show the positions of the bar before deformation and the solid lines, its positions after deformation. The length of the bar

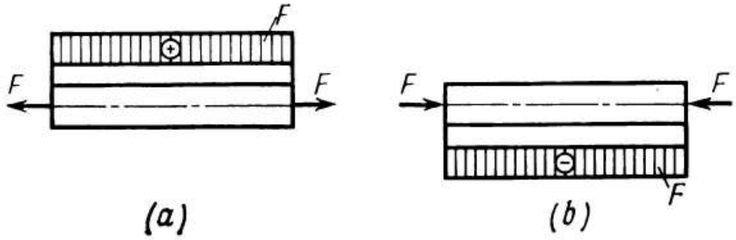


Fig. 75.
Diagrams of longitudinal forces in (*a*) tension and (*b*) compression

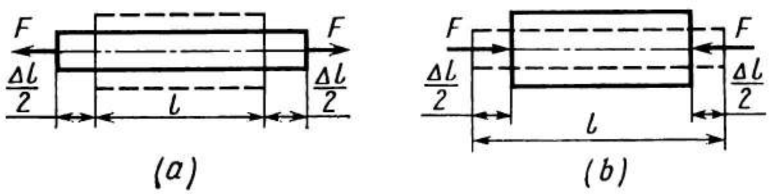


Fig. 76.
A bar before and after loading in (*a*) tension and (*b*) compression

after tension becomes equal to $l + \Delta l$ and after compression, $l - \Delta l$.

If a number of forces act on a bar, the internal force factor, i.e. the longitudinal force N, can also be found by the method of sections. In that case, the rule of signs is applied to the active forces and, using the conditions of equilibrium, the force N in the section considered is determined as the algebraic sum of all forces acting on the bar up to the section being considered:

$$N = \sum_{i=1}^{n} F_i$$

where F_i is an ith force acting on the bar.

If it turns out that $N > 0$, it will be directed away from the section and is a tensile force; otherwise, it is directed into the section and is a compressive force.

Let us now go over to determining the stresses. In either tension or compression of a bar, an internal force factor acts on the section considered; this is a longitudinal force perpendicular to the plane of section of the bar. Therefore, according to the hypothesis of plane sections, a normal stress is distributed uniformly over the bar

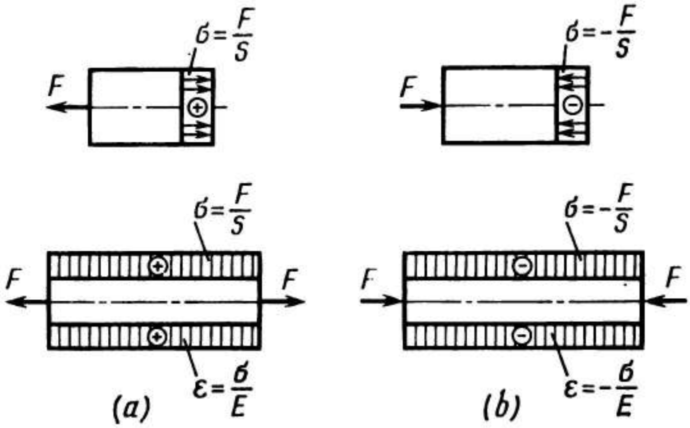


Fig. 77.
Diagrams of normal stresses and strains in a bar in (*a*) tension and (*b*) compression

section (Fig. 77) and, following the definition of stress, it may be established that a normal stress appears in the bar section:

$$\sigma = \frac{N}{S}$$

where N is the longitudinal force, and S is the cross-sectional area of the bar.

Stresses are measured in pascals (Pa) or megapascals (MPa). It should be remembered that 1 MPa $=$ 10^6Pa $=$ $=$ 1 N/mm^2 $=$ 10 kgf/cm^2.

The sign of a stress is decided by the longitudinal force. The strain is found on the basis of Hooke's law:

$$\varepsilon = \frac{\sigma}{E}$$

where σ is the normal stress, and E is the modulus of elasticity in tension, or Young's modulus. Diagrams of normal stresses and strains are shown in Fig. 77.

Using Hooke's law, it is possible to find the elongation of a bar. This can be done by substituting the expressions of stress and strain into the formula of Hooke's law:

$$\sigma = \frac{N}{S}$$

$$\varepsilon = \frac{\Delta l}{l}$$

After certain transformations, we obtain:

$$\Delta l = \frac{Nl}{ES}$$

The sign of Δl is determined by the sign of force N.

As follows from this formula, the elongation of a bar is directly proportional to the force applied and the length of the bar, and inversely proportional to the modulus of elasticity E and the cross-sectional area S of the bar.

The product ES is called the stiffness of the bar cross section. With a higher stiffness, the elongation of a bar is smaller. Owing to elongations, various sections of a bas are displaced by a different magnitude U along the axis. For instance, the section B (Fig. 78) acquires a displacement

$$U_B = U_A + U_{BA}$$

where U_A is the displacement of section A; U_{BA} is the displacement of section B relative to section A, $U_{BA} = \Delta l_{AB}$; and Δl_{AB} is the elongation of portion AB of the bar.

Consider an example for the application of the theoretical considerations given above.

Example. Construct the diagrams of longitudinal forces, stresses, strains, and displacements for a bar of a stepped profile loaded as shown in Fig. 78.

Solution. The construction of the diagram of longitudinal forces is started with division of the bar into three portions I, II, and III according to the forces applied to it. Then the internal longitudinal force in the sections of each portion is determined by the method of sections.

We have to begin with the free end of the bar, since an unknown constraint reaction acts in the other (fixed) end. For each of the sections considered, the force N is

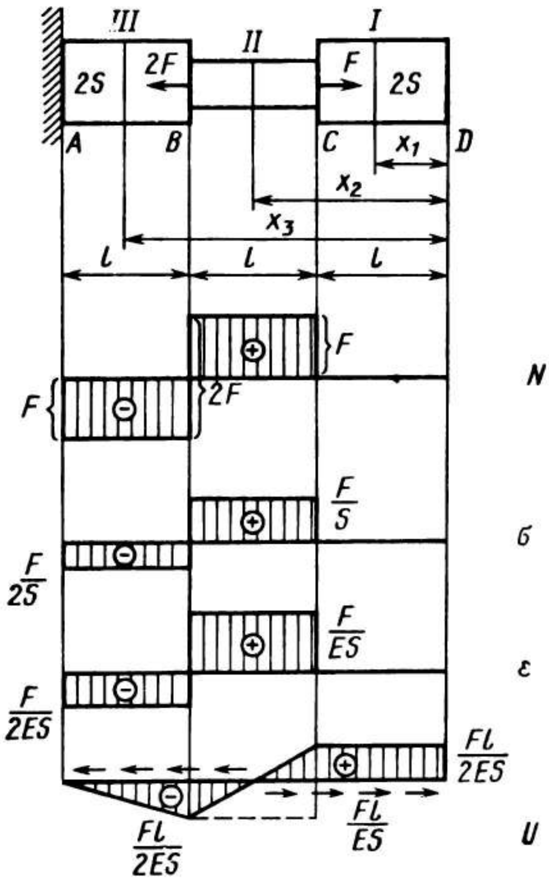


Fig. 78.
Loading scheme and diagrams of N, σ, ε, and U for a stepped bar

determined as the algebraic sum of all forces acting on the bar up to that section (Fig. 79):

in portion I

$$N_1 = 0$$

in portion II

$$N_2 = F$$

and in portion *III*

$$N_3 = F - 2F = -F$$

The minus sign indicates that the force N_3 is directed into the section considered and is compressive. The diagram of force N is shown in Fig. 78. One should learn the important rule for constructing the diagrams of longitudinal forces, since similar rules will be often used in the

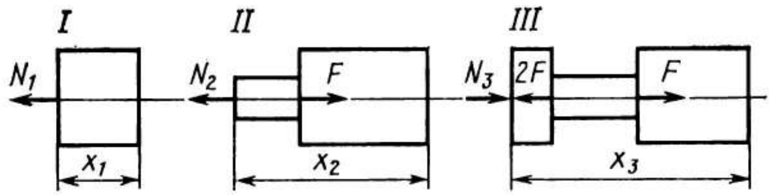


Fig. 79.
Determination of force N in three sections of a bar

later discussion. In points where a longitudinal force is applied, including support reactions, a diagram of longitudinal forces exhibits a jump equal in magnitude to the force applied. In our example, there are three forces (including the support reaction) and the diagram of forces has respectively three jumps equal in magnitude to these forces. Following this rule, a diagram of longitudinal forces can be constructed more easily, without setting up the equilibrium conditions for each portion of the bar.

We begin the construction of the diagram of longitudinal forces with the free end of the bar (see Fig. 78). There are no forces in portion *I*, because of which the longitudinal force is equal to zero. On moving to the beginning of portion *II*, where the force F is applied, the diagram must have a jump equal to that force. Since the force F is directed away from the section considered, it is positive, i.e. is a tensile force, and its magnitude should be laid off upward on the diagram. At the end of portion *II*, in the section where the second force, $2F$, is applied, the diagram will have another jump by the magnitude of $2F$. Since this force is directed towards the cross section and is negative, the jump in the N diagram should

be downwards by the value of $2F$. The third jump corresponds to the reaction force F in the built-in end of the bar.

The diagram of stresses σ can be easily recalculated from the diagram of forces by dividing the values of N by the corresponding area of bar section. It is essential to consider the sign of force (see Fig. 78). Similarly, the diagram of relative deformations ε is constructed by dividing each stress σ by the modulus of elasticity E in accordance with Hooke's law (see Fig. 78).

The displacements are determined beginning with the bar section at the built-in end where the bar displacement is evidently zero. Then, the elongation of each portion of the bar can be found by the formula:

$$\Delta l = \frac{Nl}{ES}$$

where N, l, S, and E are taken for each portion by considering properly the sign of N. After that the displacements U can be found. We then obtain for the end of portion III, the displacement of section B:

$$U_B = \Delta l_3 = -\frac{Fl}{2ES}$$

where Δl_3 is the elongation of portion III. For the end of portion II (section C) we have:

$$U_C = U_B + \Delta l_2 = -\frac{Fl}{2ES} + \frac{Fl}{ES} = \frac{1}{2}\frac{Fl}{ES}$$

where $\Delta l_2 = Fl/ES$ is the elongation of portion II.

The total elongation of the bar is the algebraic sum of the elongations of all portions:

$$\Delta l = \Delta l_1 + \Delta l_2 + \Delta l_3 = 0 + \frac{Fl}{ES} - \frac{Fl}{2ES} = \frac{Fl}{2ES}$$

The diagram of displacements U is shown in Fig. 78, where the directions of displacements of bar sections are indicated by arrows.

For estimating the strength of the bar in tension or compression, it is essential to consider the evident condition of strength which states that the maximum stress

σ_{max} in any section of the bar must not exceed the allowable stress $[\sigma]$:

$$\sigma_{max} \leqslant [\sigma]$$

The section of the bar in which the stress σ_{max} is developed is the critical section.

Noting that

$$\sigma_{max} = \left(\frac{N}{S}\right)_{max}$$

and following from the condition of strength

$$\left(\frac{N}{S}\right)_{max} \leqslant [\sigma]$$

it is possible to solve three problems as follows:

(1) check the strength of a bar for the specified load and cross-sectional area:

$$\sigma_{max} = \left(\frac{N}{S}\right)_{max} \leqslant [\sigma]$$

(2) determine the cross-sectional area S for the specified load and allowable stress $[\sigma]$:

$$S \geqslant \frac{N}{[\sigma]}$$

(3) determine the allowable load on a bar by the specified cross-sectional area and allowable stress:

$$[F] \leqslant [\sigma] S$$

Approximate values of allowable stresses for selected materials are given in Table 3.

Example. Select the cross-sectional area for the bar considered in the previous example for the following initial data: $F = 100$ kN and $[\sigma] = 160$ MPa.

Solution. The smallest cross-sectional area of the bar is chosen from the condition of strength for the critical section:

$$S = \frac{N}{[\sigma]}$$

Table 3

Material	Allowable stress, MPa	
	tensile $[\sigma_t]$	compressive $[\sigma_c]$
Grey cast iron:		
grade SCh10	20-30	70-110
grade SCh15	25-40	90-150
grade SCh20	35-55	160-250
Structural carbon steel	60-250	60-250
Structural alloy steel	100-400	100-400
Duralumin	80-150	80-150
Brass	70-140	70-140
Concrete	0.1-0.7	1-9
Fabric-based laminate	15-30	30-40

Noting that 1 MPa = 1 N/mm², we express the force in newtons and find the smallest cross-sectional area in square millimetres:

$$S = \frac{100 \times 10^3}{160} = 625 \text{ mm}^2$$

With the known cross-sectional area, the cross-sectional dimensions of the bar are chosen so as to obtain the required shape (a circle, square, etc.). If, for instance, the bar should have a square cross section with the side a, the latter is found by the formula:

$$a = \sqrt{F} = \sqrt{625} = 25 \text{ mm}$$

4.2. Shear

As distinct from tension and compression where the active force is directed along the axis of a bar, the forces active in shear are directed transversely to the bar axis (Fig. 80).

In shear, like in tension or compression, a single force factor appears in a section, but this is a lateral shearing force Q, rather than a longitudinal force (Fig. 81). This force can be found by the method of sections, a universal method in strength of materials. If a bar is cut transversely in section I-I (see Fig. 80), it can be easily found from the equilibrium conditions for its right-hand or left-hand portion that the shearing force is equal to the resultant force in magnitude and is directed oppositely (see Fig. 81):

$$Q = F$$

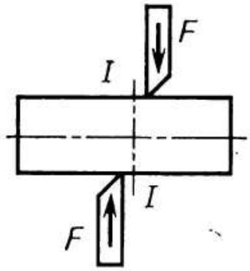


Fig. 80.
Diagram of external forces in shear

The shearing force Q is distributed uniformly over the cross section of area S and produces shear stresses τ which lie in the plane of the section:

$$\tau = \frac{Q}{S}$$

Like normal stresses, shear stresses are measured in pascals or megapascals.

The diagram of shear stresses in the section of a bar is shown in Fig. 82. The sum of all forces due to shear

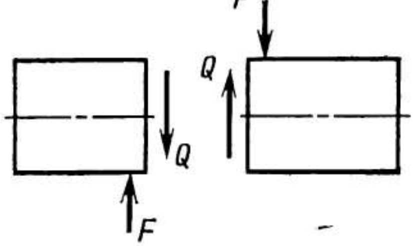


Fig. 81.
Equilibrium of external and internal forces

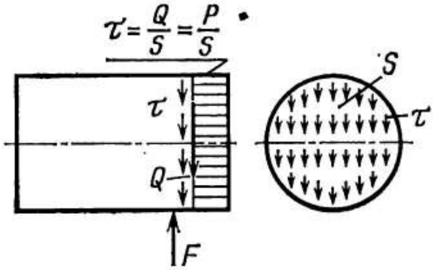


Fig. 82.
Diagram of shear stresses

stresses acting in the cross section is equal to the shearing force:

$$Q = \tau S$$

As in tension or compression, the effect of shearing forces on a bar is associated with the appearance of respective deformations.

Let us consider these deformations. The deformation of a bar on shear is shown schematically in Fig. 83. The lateral forces F shift the two sections I and II relative to each other so that section II is displaced relative to section I by a, which is called the absolute shear displacement. The absolute shear displacement depends on the distance b between the active forces F. With a larger distance b, a greater absolute displacement is produced by the same active force.

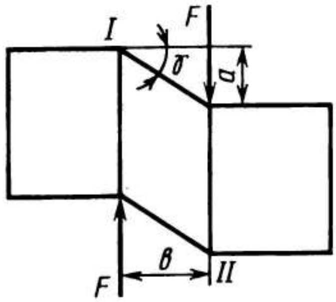


Fig. 83.
Shear deformation of a bar

There is a formal analogy between shear and tension: the elongation Δl in tension is also dependent on the length of a bar. Therefore, as in tension, we can go over to a dimensionless quantity, i.e. angular displacement (or shift) which is denoted by the Greek letter γ:

$$\gamma = \frac{a}{b}$$

where a is the absolute displacement, and b is the distance between the shearing forces.

Let us recall that an analogous dimensionless quantity, strain, has been taken in the analysis of tension and compression:

$$\varepsilon = \frac{\Delta l}{l}$$

As may be seen from Fig. 83, angular shift is a measure of shear deformation and is equal to the tangent of shear angle or, since the latter is small, to the angle proper:

$$\frac{a}{b} = \tan\gamma \cong \gamma$$

As has been found experimentally, there is a certain proportionality between shear stresses τ and angular shifts γ, which exists up to a definite limit of stress (similar to the proportionality between stresses σ and strains ε in tension):

$$\tau = G\gamma$$

where G is the tangential modulus of elasticity, or shear modulus.

This relationship is called Hooke's law in shear. Like Hooke's law in tension, it is valid within elastic deformations of a bar.

The elastic modulus and shear modulus of a particular material are interrelated as follows:

$$G = \frac{E}{2(1+\mu)}$$

where μ is Poisson's ratio.

The shear modulus G, like the elastic (Young's) modulus, E, is a property of the material of a bar and has a definite value for each material. The shear modulus is also measured in pascals or megapascals. The shear moduli G, MPa, of selected materials are given below:

Steel	0.8×10^5	Glass-reinforced plastic	0.08×10^5
Copper	0.45×10^5	Rubber	4
Aluminium	0.26×10^5		

In shear strength calculations, it is required that the active stresses be not higher than an allowable stress:

$$\tau = \left(\frac{Q}{S}\right) \leqslant [\tau]$$

where $[\tau]$ is the allowable shear stress.

For rough calculations it may be taken that

$$[\tau] = 0.5\,[\sigma]$$

where $[\sigma]$ is the allowable tensile stress.

Using the condition of strength, the following problems on shear can be solved:

(1) estimate the strength for the specified loads F and cross sectional area S:

$$\tau = \frac{Q}{S} \leqslant [\tau]$$

(2) determine the cross-sectional area for the specified loads and allowable stresses:

$$S = \frac{Q}{[\tau]}$$

(3) determine the allowable load $[F]$ for the specified allowable stresses and cross-sectional area:

$$[F] = S\,[\tau]$$

The examples that follow will demonstrate the practical application of the theory.

Example 1. Calculate the shear strength of duralumin rivets of a diameter of 1 cm which join two metal sheets

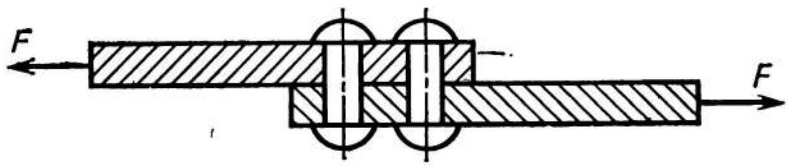


Fig. 84.
A riveted joint

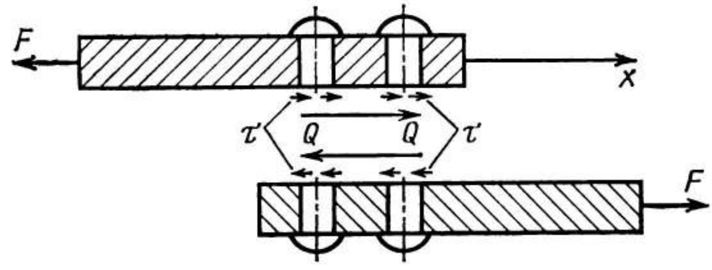


Fig. 85.
Diagram of external and internal forces acting in a riveted joint

loaded by tensile forces. The number of rivets $n = 2$ and the active force $F = 4$ kN (Fig. 84).

Solution. The forces F which tension the two steel sheets act transversely to the axes of rivets and produce shearing forces Q and shear stresses τ in the rivets (Fig. 85). As follows from the condition of equilibrium, the

shearing force Q is equal to the active tensile force F. Projecting these forces onto the x axis, we have:

$$+Q - F = 0, Q = F$$

Assuming that shear stresses are distributed uniformly, we find:

$$\tau = \frac{Q}{nS} = \frac{F}{n\frac{\pi d^2}{4}} = \frac{4F}{\pi d^2 n}$$

where n and d are the number and diameter of rivets, and S is the cross-sectional area. With a greater number of rows of rivets, stress distribution between them becomes substantially non-uniform. The extreme rows of rivets are overstressed, whereas those in the mid are understressed. Non-uniform stressing of rivets can be accounted for by introducing correction factors, but for rough estimation of strength it is possible to use the method described above. Let us now determine the numerical values of shear stresses:

$$F = 4 \text{ kN} = 4 \times 10^3\text{N}, d = 1 \text{ cm} = 10 \text{ mm},$$

$$\tau = \frac{4F}{\pi d^2 n} = \frac{4 \times 4 \times 10^3}{\pi \times 10^2 \times 2} = 25 \text{ MPa}$$

The allowable shear stress

$$[\tau] = 0.5 [\sigma]$$

We find in Table 3 for duralumin: $[\sigma] = 100$ MPa. Then

$$[\tau] = 0.5 \times 100 = 50 \text{ MPa}$$

Thus, $\tau \leqslant [\tau]$ and the shear strength condition for rivets is satisfied.

Example 2. Determine the length of a welded joint of two steel sheets of a thickness $\delta = 0.5$ cm and width $a = 10$ cm which are tensioned by forces $F = 25$ kN (Fig. 86).

Solution. We assume for the calculation that shear stresses are distributed uniformly in the welded joint, i.e.:

$$\tau = \frac{F}{S}$$

The welded joint shown in the figure is a typical lap joint consisting of two end lap welds *1* (with the area S_e)

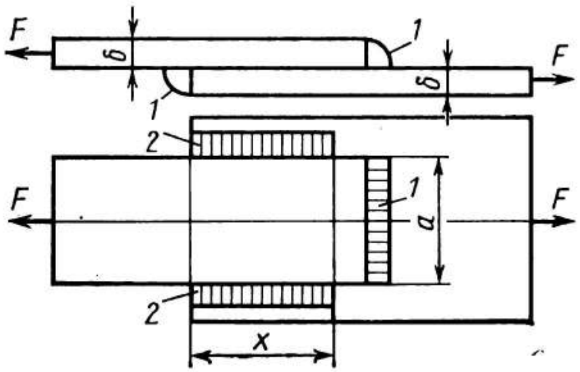


Fig. 86
Loading diagram for a welded joint

and two side lap welds *2* (with the area S_s). Thus, the total area of shearing is:

$$S = S_e + S_s$$

End lap welds are usually made to the full width of sheets, while the length of side lap welds is determined by the condition of shear strength of a welded joint:

$$\tau = \frac{F}{S_e + S_s} \leqslant [\tau]$$

where $S_e = 2a \times 0.7\delta$ is the total area of shear of two end lap welds, and $S_s = 2x \times 0.7\delta$ is the total area of shear of two side lap welds, 0.7δ is the minimal size of the cross section of a weld measured along the bisector of right angle (Fig. 87). Thus, we obtain after substitution:

$$\frac{F}{2a \times 0.7\delta + 2x \times 0.7\delta} = [\tau], \quad x = \frac{F}{1.4\delta\,[\tau]} - a$$

Substituting the initial values reduced to the same units, we get:

$$x = \frac{25 \times 10^3}{1.4 \times 5 \times 30} - 100 = 19 \text{ mm}$$

where $[\tau] = 0.5\,[\sigma] = 0.5 \times 60 = 30$ MPa (for structural carbon steel according to Table 3). If x has turned

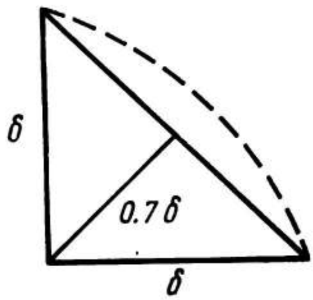


Fig. 87.
Calculation of the minimal cross-sectional area of a weld

out to be negative, this means that the end lap welds alone ensure the required strength of the welded joint.

4.3. Torsion

A bar subjected to torsion is called a shaft. In torsion of a shaft, there appears a single internal force factor, a torsional moment, or torque, which acts in the plane of shaft cross section.

As in the earlier discussion, let us use the method of sections for determining the internal torsional moment (Fig. 88). The rule of signs for torque will be assumed according to the data of Chapter one: if viewed from the end of a shaft the latter rotates counter-clockwise, the torque on the shaft is considered to be positive.

The internal torque appearing in section I, M_t^{in}, can be found from the condition of equilibrium for the right-hand or left-hand portion of the twisted shaft:

$$-M_t + M_t^{in} = 0$$

whence

$$M_t^{in} = M_t$$

In further discussion, the subscript 'in' of torsional moments will be omitted.

As in tension, compression and shear, the internal force factors, i.e. the torsional moment in this case, has

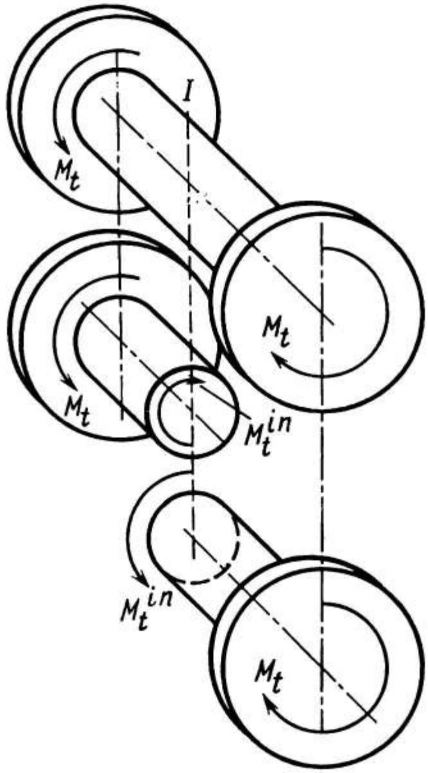


Fig. 88.
Determination of internal torsional moment by the method of sections

mutually opposite directions for the right- and left-hand portions of the shaft in section *I*. Let us now go over from the three-dimensional graphical presentation of torsion of the shaft to a conditional planar presentation (Fig. 89). In this figure, the torsional moment is depicted by two circles: a circle with a point inside implies that the motion is towards the observer and a circle with the plus sign implies the motion away from the observer.

A graph of distribution of internal torsional moments along the length of a shaft is called the torque diagram. The torque diagram for the shaft considered is shown in

Fig. 89. In any section of the shaft there acts a constant torsional moment, as we have been convinced by taking an arbitrary section I. The negative sign of the diagram of torsional moments M_t^- is determined by the fact that the moment acts clockwise if viewed from the shaft end. Consider the principle of construction of the torque diagram taking a more intricate example of a shaft acted upon by a number of torsional moments (Fig. 90). The shaft is in equilibrium under the action of torsional moments M_{t1}, M_{t2}, M_{t3}, i.e. the algebraic sum of the torsional moments is equal to zero:

$$-M_{t1} + M_{t2} - M_{t3} = 0$$

Let us check the equilibrium equation by the numerical values of M_t given in Fig. 90:

$$-30 + 40 - 10 = 0$$

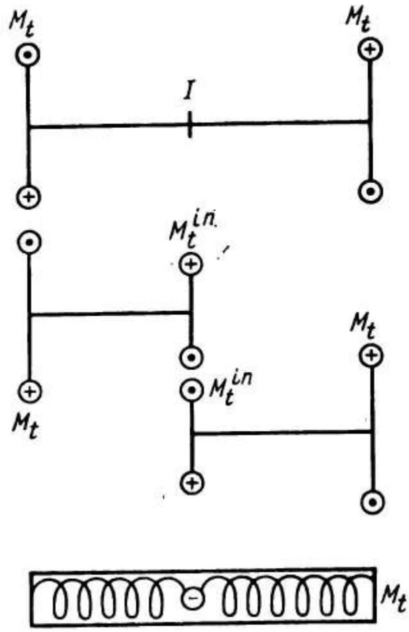


Fig. 89.
Torsional moment diagram

For constructing the torque diagram, the shaft can be divided into four portions as shown in Fig. 90 and one section in each of the portions will be considered. Each new portion of the shaft is characterized by the appearance of an additional torsional moment. The first portion includes the left-hand part of the shaft up to the point of application of torsional moment M_{t1}. In section I of the first portion, there appears no torsional moment, which is confirmed by the equilibrium conditions for the rejected portion of the shaft: $M_t^I = 0$.

The second portion of the shaft is confined between the sections of application of torsional moments M_{t1} and M_{t2}. In section II there appears an internal torsional

moment M_t^{II} which can be found from the equilibrium condition:

$$-M_{t1} - (-M_t^{II}) = 0, \qquad M_t^{II} = M_{t1}$$

By a similar procedure, we can determine the torsion-

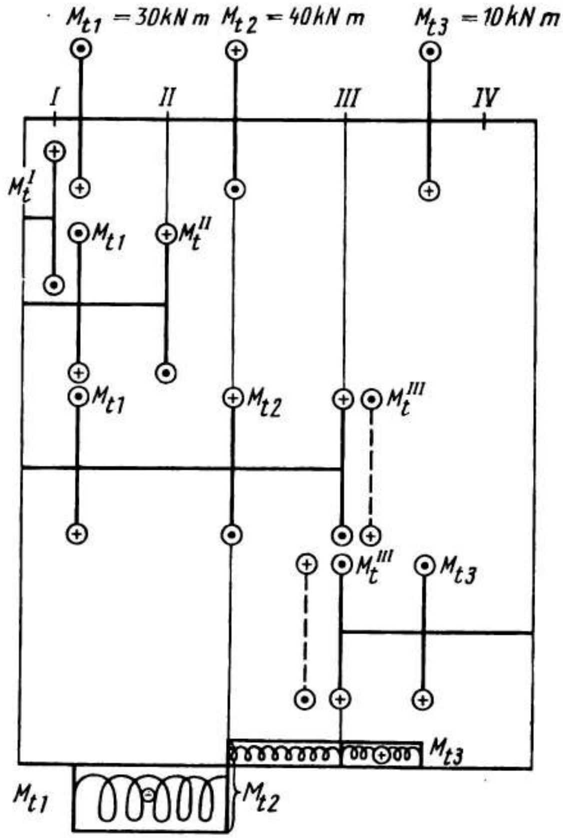


Fig. 90.
Construction of a torsional moment diagram

al moment M_t^{III} that acts in the third portion:

$$-M_{t1} + M_{t2} + M_t^{III} = 0$$

The sign of M_{t2} and M_t^{III} is opposite to that of M_{t1}, since these moments are directed oppositely to M_{t1}:

$$M_t^{III} = M_{t1} - M_{t2} = 30 - 40 = -10 \text{ kN m}$$

Thus, M_t should be directed oppositely to what is shown in Fig. 90. The actual direction is indicated by the dotted line; it corresponds to the positive direction of the moment, i.e. counter-clockwise.

In order to check that the internal torsional moments have been found correctly, it would be useful to consider the equilibrium of the right-hand portion of the shaft cut in section *III*:

$$-M_{t3} - M_t^{III} = 0; \quad M_t^{\mathrm{III}} = -M_{t3} = -10 \text{ kN m}$$

The minus sign indicates that the real torsional moment in section *III* acts in the opposite direction.

The example on determining the torsional moments in various sections of a shaft, which we have calculated above (see Fig. 90), suggests the following conclusion: the internal torsional moment in a particular section of a shaft is equal to the algebraic sum of all external torsional moments which act on the shaft up to the section considered:

$$M_t = \sum M_{t\,i}$$

Let us now construct the diagram of torsional moments. For each portion of the shaft, we lay off M_t on a chosen scale much in the same way as has been done for constructing the diagram of longitudinal forces N in tension. The general rule for construction of diagrams of torsional moments M_t is similar to that for construction of diagram of longitudinal forces N: a diagram of torsional moments has jumps in points of application of torsional moment, which are equal to the magnitude of the torsional moment applied. In the case considered, when moving along the shaft from the left to right, we observe the first jump equal to M_{t1}, the second jump equal to M_{t2}, and the third jump M_{t3}. The direction of a jump is associated with the direction and sign of the torsional moment. The moment M_{t1} is negative and the jump corresponding to it is directed downward, whereas the moments M_{t2} and M_{t3} are positive and their jumps are directed upward.

Let us now determine the stresses appearing in a shaft in torsion. Since the sole internal force factor in

torsion, i.e. the torsional moment, acts in the plane of shaft cross section, the stresses produced by it also lie in that section and are called tangential stresses τ.

We have to establish the correlation between tangential stresses and torsional moments. This can be done by considering the following experiment. Let a square network be applied onto the outside surface of a round shaft (Fig. 91*a*). Suppose that the shaft is rigidly fixed at one

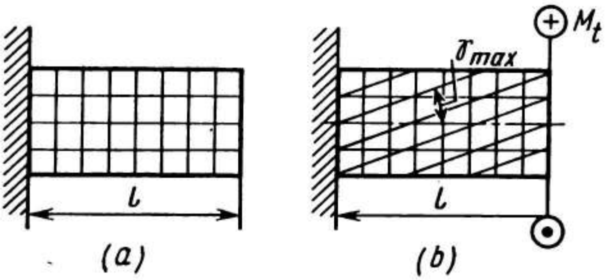


Fig. 91.
Deformation of a shaft in torsion

end and is twisted by the moment M_t at the other (Fig. 91*b*). The following picture will then be observed on the application of the moment:

(a) the square network on the shaft surface changes to a rhombic, i.e. each square deforms similarly to the shear deformation of the bar considered in the previous section of the book;

(b) the total length l of the shaft has not changed, since there are no longitudinal forces which could deform the shaft in the axial direction;

(c) the end of the shaft, which was flat before the deformation, has remained flat upon application of the load; and

(d) the radius of the shaft cross section has not changed on torsion.

This experiment is helpful for revealing the mechanism of deformation on a shaft in torsion. First, the familiar hypothesis of plane sections turns out to be valid for torsion too. As may be seen, all sections of the shaft remain plane after torsional deformation and only turn relative

to one another. Thus, there are shear deformations between the sections, which have been discussed in Sec. 4.2.

Let us consider a shaft under the action of a torsional moment (Fig. 92) and introduce a number of definitions.

As a shaft of length l is twisted by a torsional moment M_t, the free end of the shaft turns through the to-

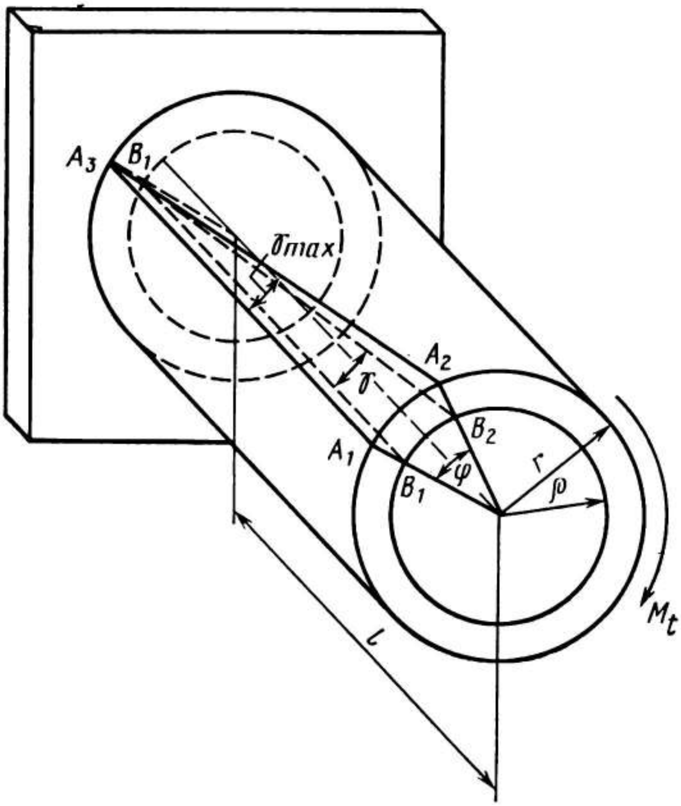


Fig. 92.
Calculation of geometrical parameters for a shaft in torsion

tal twisting angle φ. In this process, as follows from the experiment discussed above, the generatrices of the shaft surface acquire the shape of helical lines with the angle of incline γ_{max}. Then, a point A_1 will be transferred upon torsional deformation into point A_2. The length of the arc A_1A_2 can be found by two methods;

$$A_1A_2 = \gamma_{max}l$$
$$A_1A_2 = \varphi r$$

Thus, we obtain the relationship between the absolute twisting angle φ and the maximum shear angle γ_{max}:

$$\gamma_{max} l = \varphi r$$

In twisting, the points of the shaft surface acquire the maximum displacement, and therefore, the maximum relative shear angles γ_{max}.

Let us now analyse how internal points of the shaft are displaced on twisting. Considering the surface of an internal cylinder of radius ρ (see Fig. 92), we may see that point B_1 has been transferred under the action of the torsional moment M_t into point B_2 and the corresponding shear angle on the surface of the cylinder of radius ρ will be γ. Then, similarly to what has been done above, we obtain:

$$B_1B_2 = \gamma l$$
$$B_1B_2 = \varphi\rho$$

whence

$$\gamma l = \varphi\rho, \qquad \gamma = \frac{\varphi}{l}\rho$$

To go over to stresses, let us use Hooke's law in shear and substitute into it the expression for γ:

$$\tau = \gamma G = \frac{\varphi}{l} G\rho$$

If an elementary area dS at a distance ρ from the centre of gravity of the section is acted upon by the stress τ, the elementary torsional moment will be equal to the elementary force $\tau\, dS$ multiplied by the arm ρ (Fig. 93):

$$dM_t = \tau\, dS\rho$$

The total moment can be obtained by adding the elementary moments over the area of the section; it is equal to the torque acting in that section:

$$M_t = \int_S \tau\rho\, dS = \frac{\varphi G}{l} \int_S \rho^2\, dS$$

It may be recalled that the integral obtained

$$\int_S \rho^2 \, dS = J_p$$

is called the polar moment of inertia. The expression for M_t can then be rewritten as follows:

$$M_t = \frac{\varphi G J_p}{l}$$

whence we obtain the formula of the total twisting angle:

$$\varphi = \frac{M_t l}{G J_p}$$

This formula is similar in its structure to the formula for elongation of a bar in tension: $\Delta l = Nl/ES$, if we

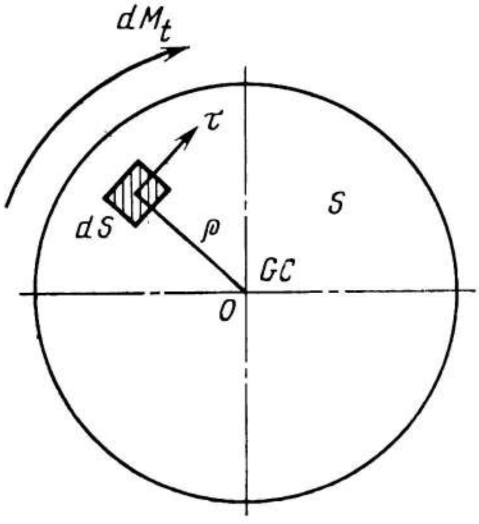


Fig. 93.
Determination of elementary torsional moment in a section of a shaft

consider that the force is replaced by torsional moment and the rigidity of a section in tension, ES, is replaced by the rigidity in torsion, GJ_p. Thus, the polar moment of inertia together with the shear modulus characterizes the rigidity of the section of a bar in torsion. By analogy with tension and shear, the concept of the relative twist-

ing angle θ can be introduced for torsion. This angle is independent of the shaft length:

$$\theta = \frac{\varphi}{l} = \frac{M_t}{GJ_p}$$

Substituting for φ/l in the formula of tangential stress τ, we obtain after cancelling:

$$\tau = \frac{\varphi}{l} G\rho = \frac{M_t}{J_p} \rho$$

As may be seen from this formula, tangential stress is directly proportional to the radius of a section. Let

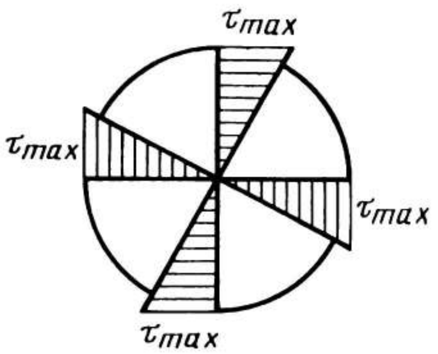


Fig. 94.
Diagram of tangential stresses

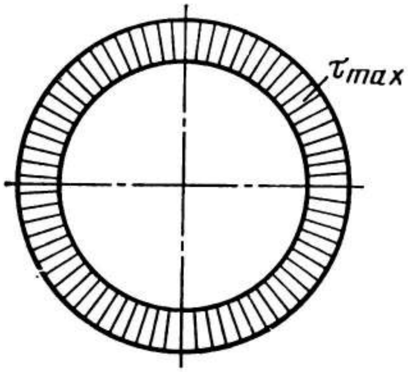


Fig. 95.
Diagram of maximal tangential stresses

us construct the diagram of tangential stresses in the section of a shaft (Fig. 94). The maximum tangential stresses act all over the external surface of the shaft; their diagram is shown in Fig. 95:

$$\tau_{\max} = \frac{M_t}{J_p} r = \frac{M_t}{J_p/r} = \frac{M_t}{W_r}$$

where $W_r = J_p/r$ is the moment of resistance in torsion. For round sections of diameter D:

$$J_p \approx 0.1D^4, \text{ and } W_r \approx 0.2D^3$$

and for annular sections

$$J_p \approx 0.1\,(D^4 - d^4), \quad \text{and} \quad W_r = 0.2D^3\,(1 - C^4)$$

where $C = d/D$ (D is the outside diameter of a section, and d is its inside diameter).

Torsional deformation of bars of non-circular, in particular of rectangular, cross section is characterized in that the cross sections of the bar do not remain plane, but are curved beyond the cross-sectional plane (Fig. 96). This is an evidence that longitudinal axial forces

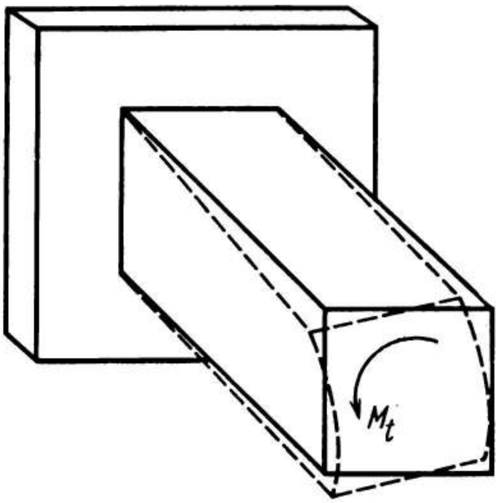


Fig. 96. Twisting of a rectangular bar

appear in the sections in addition of torsional moments. As a result, the problem of stress determination becomes more complicated than for round shafts, for which the hypothesis of plane sections is valid. The problem can be solved by the methods of the theory of elasticity.

Let us give, without proof, the final results of determining the maximum tangential stresses in torsion of rectangular cross-sectional bars, τ_{max}, and the total twisting angle φ:

$$\tau_{max} = \frac{M_t}{W_r} \quad \text{and} \quad \varphi = \frac{M_t l}{GJ_p}$$

where $W_r = \alpha ab^2$ is the moment of resistance to torsion of a rectangular bar with the sides a and b; $J_p = \beta ab^3$ is the polar moment of inertia of the rectangular bar;

and α and β are coefficients whose values, depending on the side ratio of rectangle, are as follows:

a/b	1	2	3	4	5	10	20	90
α	0.21	0.25	0.27	0.28	0.29	0.31	0.32	0.33
β	0.14	0.23	0.26	0.28	0.29	0.31	0.32	0.33

For thin bars in which one side of the rectangular cross section exceeds the other more than 5 times, it may be taken that $\alpha = \beta \approx 0.3$.

The diagram of tangential stresses in a cross section and on the surface of a rectangular bar twisted by mo-

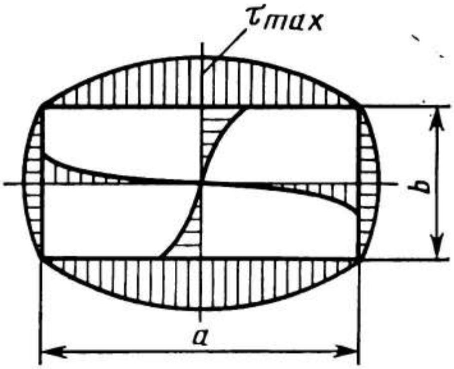


Fig. 97.
Stress diagram for a rectangular bar

ment M_t is shown in Fig. 97. The maximum tangential stresses appear in the mid of the larger side of the rectangle. At the corners of the section, tangential stresses are equal to zero.

As in all preceding calculations, the strength condition for a shaft in torsion states that the maximum design stress in torsion of a shaft be not higher than an allowable stress:

$$\tau_{\max} = \frac{M_t}{W_r} \leqslant [\tau]$$

In rough calculations, the allowable tangential stress can be determined by the formula for shear stresses:

$$[\tau] = 0{,}5\,[\sigma]$$

where $[\sigma]$ is the allowable stress in tension (to be taken from Table 3).

The strength condition makes it possible to solve three types of problems:

(1) estimate the strength of a shaft for the known load and dimensions:

$$\tau_{max} \leqslant [\tau]$$

(2) determine the required moment of resistance in torsion by the known allowable stress $[\tau]$ and load M_t:

$$W_r = \frac{M_t}{[\tau]}$$

(3) determine the allowable load on a shaft by the known allowable stress $[\tau]$ and geometrical dimensions of the shaft section:

$$[M_t] = [\tau]\, W_r$$

Shafts of an appreciable length must additionally satisfy the rigidity condition, i.e. the maximum relative twisting angle θ must not exceed the allowable relative twisting angle $[\theta]$:

$$\theta = \frac{M_t}{GJ_p} \leqslant [\theta]$$

where $[\theta]$ is the allowable relative twisting angles in radians per metre of the shaft length.

The allowable relative twisting angle is usually specified in degrees per metre of length. The formula of shaft rigidity will then have a somewhat different form:

$$\theta = \frac{180 M_t}{\pi G J_p} \leqslant [\theta]$$

For shafts of moderate dimensions, the allowable relative twisting angle $[\theta]$ can be roughly taken of an order of 0.5° per metre of length. Like the condition of strength, the condition of rigidity makes it possible to solve three types of problems on determining θ, J_p or $[M_t]$ by known initial data.

Example 1. Construct the diagrams of torsional moments, relative and absolute twisting angles, and maximum tangential stresses along the length of a round steel

shaft of a diameter $D = 10$ cm. Determine the allowable torsional moment from the strength conditions of the shaft. The allowable tangential stress $[\tau] = 50$ MPa. Check the shaft for rigidity (Fig. 98).

Solution. Let us start with constructing the diagram of torsional moments. The values of θ and τ_{max} can be found by substituting the initial values of the variable parameters into the corresponding formulae. Let us divide the shaft into four portions (see Fig. 98). The construc-

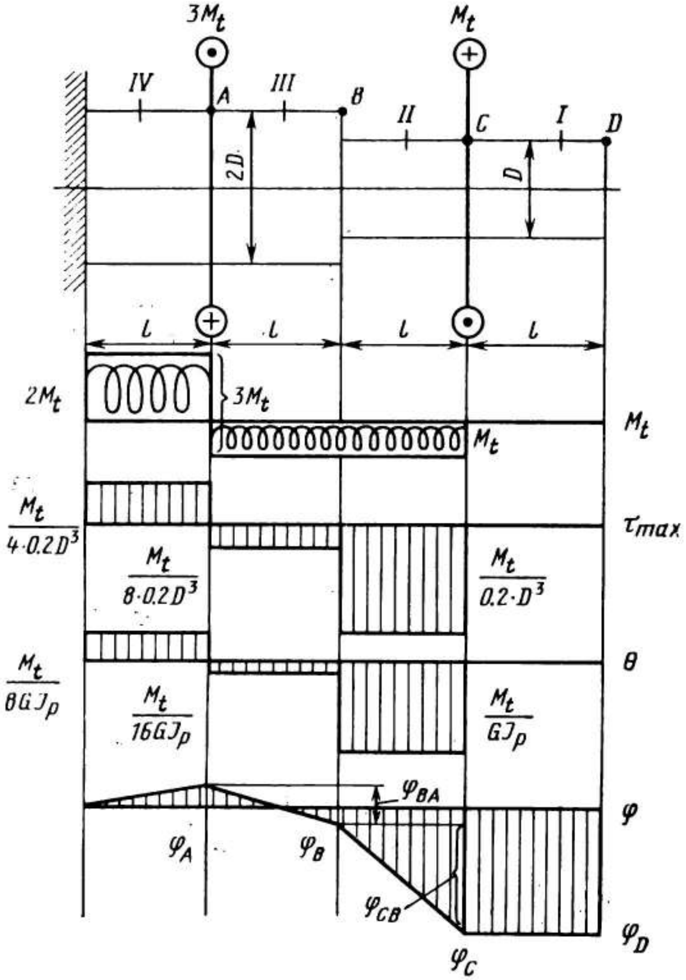


Fig. 98.
Calculation of a round shaft for torsion

tion of the diagram of torsional moments we begin with the free end, since the fixed end of the shaft is loaded by an unknown torsional moment of the support. We determine the torsional moment in portion I from the equilibrium condition by the method of sections: $M_t^I = 0$, and therefore, $\theta = 0$ and $\tau_{\max} = 0$. For portion II, we find:

$$M_t^{II} = -M_t$$

$$\theta = \frac{M_t^{II}}{GJ_p^{II}} = -\frac{M_t}{0.1GD^4} = -\frac{M_t}{GJ_p}$$

$$\tau_{\max} = \frac{M_t}{W_r^{II}} = -\frac{M_t}{0.2D^3} = -\frac{M_t}{W_r}$$

The moment M_t has the negative sign, since it acts clockwise. The signs of θ and $\tau_{\max}$ are determined by the sign of M_t.

For portion III

$$M_t^{III} = -M_t$$

$$\theta = \frac{M_t^{III}}{GJ_p^{III}} = -\frac{M_t}{0.1G(2D)^4} = -\frac{1}{16}\frac{M_t}{GJ_p}$$

$$\tau_{\max} = \frac{M_t^{III}}{W_r^{III}} = -\frac{M_t}{0.2(2D)^3} = -\frac{1}{8}\frac{M_t}{W_r}$$

where $J_p = 0.1D^4$ and $W_r = 0.2D^3$.

For portion IV

$$M_t^{IV} = 3M_t - M_t = 2M_t$$

The sign of M_t^{IV} indicates that the torsional moment in portion IV is directed opposite to the torsional moment in portion III and noting that $M_t^{III} < 0$, we find that $M_t^{IV} > 0$, and therefore, θ and $\tau_{\max}$ are also positive:

$$\theta = \frac{2M_t}{0.1G(2D)^4} = \frac{1}{8}\frac{M_t}{GJ_p}$$

$$\tau_{\max} = \frac{2M_t}{0.2(2D)^3} = \frac{1}{4}\frac{M_t}{W_r}$$

Let us now analyse the absolute twisting angles of the shaft, starting with the fixed section where the twisting angle is zero.

The twisting angle at the end of portion IV, i.e. in section A relative to the fixed section O, can be found by the formula:

$$\varphi_A = \frac{M_t^{IV} l}{G J_p^{IV}}$$

or, after substitution for M_t^{IV} and J_t^{IV}, we have for portion IV:

$$\varphi_A = \frac{2M_t l}{0.1G(2D)^4} = \frac{1}{8}\frac{M_t l}{GJ_p}$$

The twisting angle of section B relative to section A

$$\varphi_{BA} = \frac{M_t^{III} l}{G J_p^{III}} = \frac{(-M_t)\, l}{0.1G(2D)^4} = -\frac{1}{16}\frac{M_t l}{GJ_p}$$

Thus, the total twisting angle of section B is the algebraic sum of the twisting angle in section A and the twisting angle of section B relative to section A:

$$\varphi_B = \varphi_A + \varphi_{BA} = \frac{1}{8}\frac{M_t l}{GJ_p} - \frac{1}{16}\frac{M_t l}{GJ_p} = \frac{1}{16}\frac{M_t l}{GJ_p}$$

Section C is twisted relative to section B by the angle:

$$\varphi_{CB} = -\frac{M_t l}{GJ_p}$$

Section C is twisted relative to restraint by the angle:

$$\varphi_C = \varphi_B + \varphi_{CB} = \frac{1}{16}\frac{M_t l}{GJ_p} - \frac{M_t l}{GJ_p} = -\frac{15}{16}\frac{M_t l}{GJ_p}$$

Since there is no moment M_t in portion I, we have

$$\varphi_{DC} = 0$$

Then the total twisting angle in section D of the shaft

$$\varphi_D = \varphi_C + \varphi_{DC} = -\frac{15}{16}\frac{M_t l}{GJ_p}$$

The minus sign indicates that the end of the shaft is twisted clockwise through an angle φ_D, when viewed from the free end. Now the diagram of twisting angles φ can be constructed (see Fig. 98).

The allowable torsional angle can be found from the condition of strength:

$$\tau_{max} = \left(\frac{M_t}{W_r}\right)_{max} \leqslant [\tau]$$

In order to determine τ_{max}, let us take two sections: section IV where the maximum torsional moment $2M_t$ is applied and the shaft diameter is $2D$, and section II where the shaft has the minimum diameter D:

$$\tau_{max}^{IV} = \frac{2M_t}{0.2\,(2D)^3} = \frac{M_t}{0.8D^3} \quad \text{and} \quad \tau_{max}^{II} = \frac{M_t}{0.2D^3}$$

As may be seen from these formulae, $\tau_{max}^{II} > \tau_{max}^{IV}$, and therefore:

$$\tau_{max} = \tau_{max}^{II} = \frac{M_t}{0.2D^3}$$

The allowable torsional moment is found from the strength condition:

$$[M_t] = 0.2D^3\,[\tau] = 0.2 \times 100^3 \times 50 = 10^7 \text{ N mm}$$

Let us now check the shaft rigidity. Determine the maximum relative twisting angle in degrees per metre of length:

$$\theta_{max} = \frac{180M_t}{\pi G J_p} = \frac{180 \times 10^7 \times 10^3}{\pi \times 0.8 \times 10^5 \times 0.1 \times 100^4} = 0.72$$

where $J_p = 0.1D^4$.

As has been given earlier, the rigidity condition for shafts is written as follows:

$$\theta_{max} \leqslant [\theta]$$

where $[\theta] = 0.5$.

Thus, the allowable torsional moment, as found from the strength condition, does not satisfy the rigidity condition and should be diminished by a factor $\theta_{max}/[\theta] = 0.72/0.5 = 1.4$.

Example 2. An electric motor of a power $N = 50$ kW and rotational frequency $n = 500$ rpm transmits torque to a shaft. Select the cross-sectional area of the shaft which is to be made of steel. Compare the consumption of metal for a solid shaft and a tubular shaft with the ratio of diameters $C = d/D = 0.9$, where d is the inside diameter and D is the outside diameter of the tube.

Solution. Let us find preliminarily the torque acting on the shaft, N m:

$$M_t = \frac{N}{\omega}$$

where N is the motor power, W, and $\omega = 2\pi n/60$ is the angular velocity of the shaft, radians.

Hence we find:

$$M_t = \frac{N \times 60}{2\pi n} = 9.5 \frac{N}{n}$$

Substituting numerical values of N and n, we get:

$$M_t = 9.5 \frac{50 \times 10^3}{500} = 950 \text{ N m} = 950 \times 10^3 \text{ N mm}$$

Let us now determine the geometrical dimensions of the shaft. The moment of resistance to torsion is found from the strength condition:

$$W_r = \frac{M_t}{[\tau]}$$

where $[\tau]$ is the allowable tangential stress, $[\tau] = 0.5\,[\sigma]$.

The allowable tensile stress $[\sigma]$ for steel can be taken from Table 3: $[\sigma] = 160$ MPa. Thus we have:

$$[\tau] = 0.5 \times 160 = 80 \text{ MPa}$$

The moment of resistance

$$W_r = \frac{950 \times 10^3}{80} = 11.9 \times 10^3 \text{ mm}^3$$

For a solid round shaft:

$$D = \sqrt[3]{\frac{W_r}{0.2}} = \sqrt[3]{\frac{11.9 \times 10^3}{0.2}} = 3.9 \times 10 \text{ mm} = 3.9 \text{ cm}$$

For a shaft of tubular (annular) cross section, $W_r = 0.2D^3(1 - C^4)$ with $C = 0.9$, and therefore:

$$D = \sqrt[3]{\frac{W_r}{0.2(1 - C^4)}} = \sqrt[3]{\frac{11.9 \times 10^3}{0.2(1 - 0.9^4)}} = 55 \text{ mm} = 5.5 \text{ cm}$$

Taking the same length for both shafts, the metal consumption will be determined by the cross-sectional area:

for the solid shaft

$$S_s = \frac{\pi D^2}{4} = \frac{\pi \times 3.9^2}{4} = 11.4 \text{ cm}^2$$

for the tubular shaft

$$S_t = \frac{\pi}{4}(D^2 - d^2)$$

where $d = CD$; then:

$$S_t = \frac{\pi D^2}{4}(1 - C^2) = \frac{\pi \times 5.5^2}{4}(1 - 0.9^2) = 4.5 \text{ cm}^2$$

Thus, as may be seen, the use of a tubular shaft instead of a solid one can save roughly 60% or metal. This

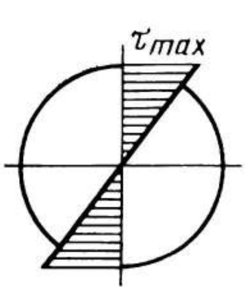


Fig. 99.
Tangential stress diagram for a solid shaft

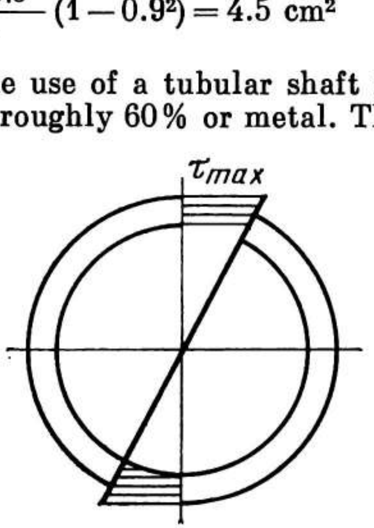


Fig. 100.
Tangential [stress (diagram for an annular section of a shaft

effect may be clear if we consider the diagrams of tangential stresses for the solid shaft (Fig. 99) and the tubular one (Fig. 100). The core portion of the solid shaft experi-

ences low stresses and is not utilized properly, because of which a tubular shaft, even of a larger diameter, gives a substantial economy in metal.

4.4. Bending

It may be briefly recalled that the main kinds of loading discussed up to this point are characterized by the appearance of a single internal force factor. It is a longitudinal force in tension or compression, lateral force in shear, and torsional moment in the cross-sectional plane in torsion.

Bending differs from these kinds of loading in that it involves two internal force factors: a lateral force Q (as in shear) and bending moment M (Fig. 101). Let us

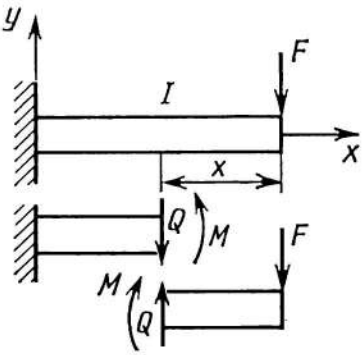


Fig. 101.
Scheme of bending

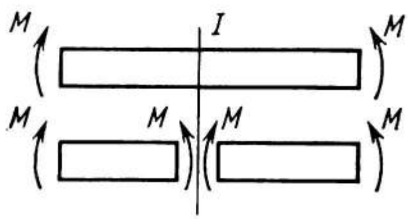


Fig. 102.
Pure bending of a bar

consider a case of bending when the plane of bending load coincides with one of the main axes of inertia of the cross section. This kind of bending is sometimes called simple bending. A bar subjected to bending is called a beam.

A particular kind of bending in which the lateral force is equal to zero is called pure bending (Fig. 102). In that case only a bending moment M appears in cross sections of a beam. As distinct from it, the kind of bending with the lateral force other than zero is termed lateral bending (see Fig. 101).

Bending is referred to simple kinds of loading, notwithstanding the fact that it involves two internal force factors: Q and M.

As will be clear from the following analysis of bending, however, the decisive internal force factor in bending is only the bending moment. The effect of lateral force in bending of beams can be neglected.

The internal force factors in bending, Q and M, can be determined by the method of sections (see Fig. 101). Let us consider the equilibrium conditions for the cut-off portion of a beam. The projections of all forces onto the y axis are equal to zero:

$$Q - F = 0; \quad Q = F$$

The moment of all forces relative to section I is equal to zero:

$$M + Fx = 0, \quad M = -Fx$$

The lateral force Q in section I is equal to the external force F that acts on the cut-off portion of the beam and the

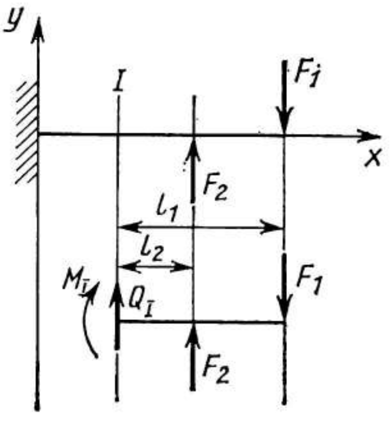


Fig. 103.
Determination of internal force factors in bending by the method of sections

bending moment M is equal to the moment of external force relative to the section considered.

Let us now consider a beam acted upon by several forces (Fig. 103). Here and further, a beam will be represented graphically by a straight line which is the locus of centres of gravity of beam sections. Let us determine the internal force factors in an arbitrary section I.

To do this, the beam is cut through in section I and the unknown internal force factors Q and M are applied in that section. The equilibrium equations for the cut-off portion of the beam are as follows:

$$\sum F_y = 0, \qquad \sum M = 0$$

The sum of the projections of all forces onto the x axis is identically equal to zero, since there are no lateral forces acting on the beam. In further discussion, the subscript 'y' in the equilibrium equations will be omitted. As follows from the first equation:

$$\sum F = -F_1 + F_2 + Q_I = 0$$

whence we find the lateral force in section I:

$$Q_I = F_1 - F_2$$

The expression just obtained can be generalized.

The lateral force in the section considered is equal to the algebraic sum of all forces acting on the beam up to the section considered:

$$Q = \sum F_i$$

Since we deal with an algebraic sum and have to consider the signs of active forces, it is essential to formulate

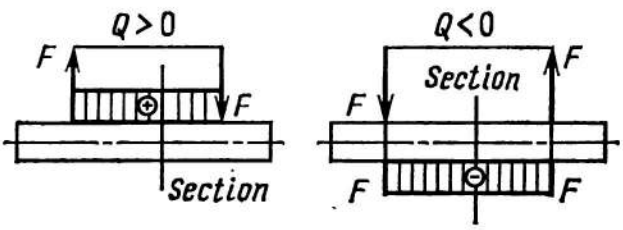


Fig. 104.
Rules of signs for a lateral force Q

the rule of signs for lateral forces in sections: the external forces (both active and reactive) lying to the left of the section in question are considered positive if they are directed upwards and negative if otherwise; the inverse is true for the forces lying to the right of the section (Fig. 104).

Let us now analyse the second equation of equilibrium. The sum of moments relative to section I is:

$$(\sum M)_I = M_1 + F_1 l_1 - F_2 l_2 = 0$$

whence

$$M_I = -F_1 l_1 + F_2 l_2$$

This expression can also be generalized. The bending moment in the section considered is the algebraic sum of the moments of all external forces relative to that section and the moments acting on the beam up to the section considered:

$$M = \sum M_i$$

The rule of signs for determining the bending moment can be formulated as follows: a moment is considered to be positive if it acts so that the beam becomes convex

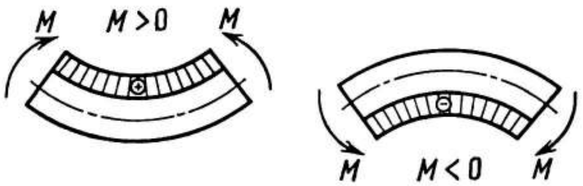


Fig. 105.
Rules of signs for a bending moment M

downwards, and vice versa (Fig. 105). This rule can be memorized more easily if it is noticed that bending moments are symbolized by two arrows which are always directed from the tensioned fibres of a bent beam to the compressed ones (see Fig. 105). Thus, a bending moment is positive if its arrows are directed upwards, and vice versa.

Kinds of supports have been described in Sec. 2.1. Let us recall that there are three main kinds of beam supports:

(1) movable hinged support (Fig. 106); (2) immobile hinged support (Fig. 107); and (3) rigidly fixed (built-in) support (Fig. 108).

The force factors that appear in supports are called support reactions. The two first kinds of supports permit

free turning of the beam, and therefore, no support moments appear in them. Only a rigidly fixed support, which does not permit the beam to turn, can create a reactive support moment.

A movable hinged support permits free axial displacement of the beam on rollers, because of which only one support reaction appears in it.

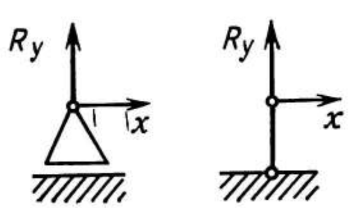


Fig. 106.
A movable hinged support

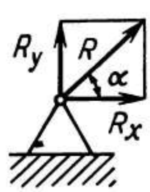


Fig. 107.
A fixed hinged support

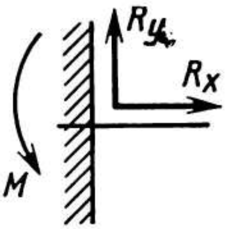


Fig. 108.
Restraint

In an immobile hinged support, the reaction is directed at an angle α to the horizontal axis; this reaction can be resolved into two component reactions along the horizontal and vertical axes. The angle α determines the ratio between the horizontal and vertical components of support reaction:

$$\tan \alpha = \frac{R_y}{R_x}$$

A rigidly fixed support gives three reactions: R_x, R_y, and M.

Consider, for example, a beam loaded by force F, which is rigidly fixed at one end and is free at the other (Fig. 109). This is what is called a cantilever beam.

The force F which acts at an angle α can be resolved along the axes x and y into the components $F \cos \alpha$ and $F \sin \alpha$ (see Fig. 109). In the built-in end of the beam, there are three unknown reactions: two forces, R_y and R_x, and moment M. Applying the unknown support reactions to the beam, we obtain the design diagram

(Fig. 110). The beam is in equilibrium, and therefore, three equilibrium equations will hold true:

$$\sum F_x = 0, \quad \sum F_y = 0, \quad \sum M = 0$$

Substituting the forces and moments acting on the cantilever beam into these equations, we get:

$$\sum F_y = R_y - F \sin \alpha = 0$$
$$\sum F_x = R_x - F \cos \alpha = 0$$
$$\sum M = M - Fl \sin \alpha = 0$$

For simplicity, the moment is taken relative to point A, so as to exclude the moments from the unknown support reactions R_x and R_y. Thus, we find that

$$R_y = F \sin \alpha; \quad R_x = F \cos \alpha; \quad M = F\ l \sin \alpha$$

As follows from the analysis of this example, if the active forces are perpendicular to the beam axis, i.e.

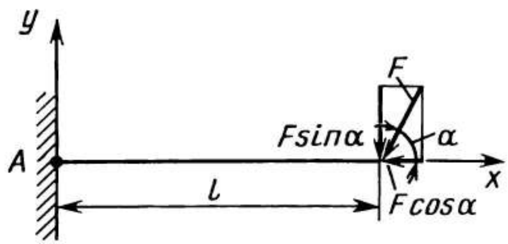


Fig. 109.
A cantilever beam under load

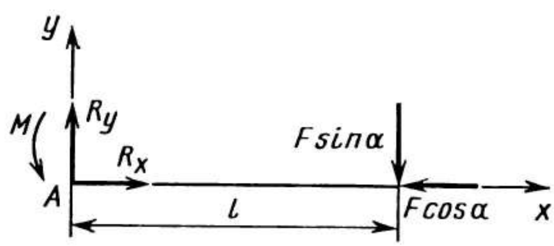


Fig. 110.
Design diagram for a cantilever beam

$\alpha = 90°$, there are no horizontal components of support reactions: $\cos \alpha = 0$, $R_x = 0$.

It is now possible to go over to the methods for constructing graphs of variation of internal force factors Q and M along the length of beams in bending, i.e. to construction of diagrams of Q and M.

Let us consider preliminarily a number of simple examples. Some of them are also given in Appendix II.

Example 1. Construct the diagrams of Q and M for bending of a beam on two supports (simply supported

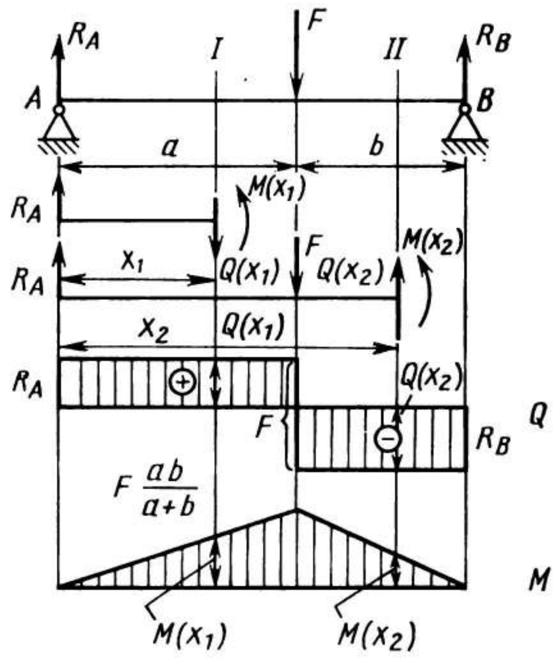


Fig. 111.
Action of a force onto a simply supported beam

beam) under the action of a concentrated force F (Fig. 111).

Solution. Let us find the support reactions R_A and R_B from the equation of equilibrium of the beam:

$$\sum M_A = Fa - R_B(a+b) = 0$$
$$\sum F = R_A - F + R_B = 0$$

It is now possible to find R_B from the first equation:

$$R_B = \frac{a}{a+b} F$$

and R_A, from the second equation:

$$R_A = F - R_B = \frac{b}{a+b} F$$

Let us divide the beam into two portions and write the expressions of lateral forces and bending moments for these portions by considering the relationships given earlier and the adopted rules of signs:

for portion I at a distance x_1 from support A

$$Q(x_1) = R_A = \frac{b}{a+b} F$$

$$M(x_1) = R_A x_1 = F \frac{bx}{a+b}$$

where $0 \leqslant x_1 \leqslant a$; with $x_1 = 0$, $M(0) = 0$; with $x_1 = a$, $M(a) = \frac{F\,ab}{a+b}$;

for portion II at a distance x_2 from support A

$$Q(x_2) = R_A - F = \frac{b}{a+b} F - F = -\frac{a}{a+b} F = -R_B$$

$$M(x_2) = R_A x_2 - F(x_2 - a) = F \frac{b}{a+b} x_2 - F(x_2 - a)$$

where $a \leqslant x_2 \leqslant a + b$; with $x_2 = a$, $M(a) = F \frac{ab}{a+b}$; with $x_2 = a + b$, $M(a + b) = 0$

Thus, in each portion of the beam, the force Q is constant and is positive for portion I and negative, for portion II. The moment depends linearly on x; it increases in portion I from 0 to $F \frac{ab}{a+b}$ and decreases from this value to zero in portion II. The diagrams of Q and M can now be constructed by considering these conclusions (see Fig. 111). It should be noted that the diagram of lateral forces has a jump in the point of application of a force, which is equal to the magnitude of that force. For instance, the Q diagram has three jumps corresponding to the forces R_A, F, and R_B.

A jump is directed upwards if the force has the plus sign and downwards if the force has the minus sign (as for force F).

Example 2. Consider bending of a simply supported beam under the action of a couple of forces or a moment M applied in the mid of the beam (Fig. 112).

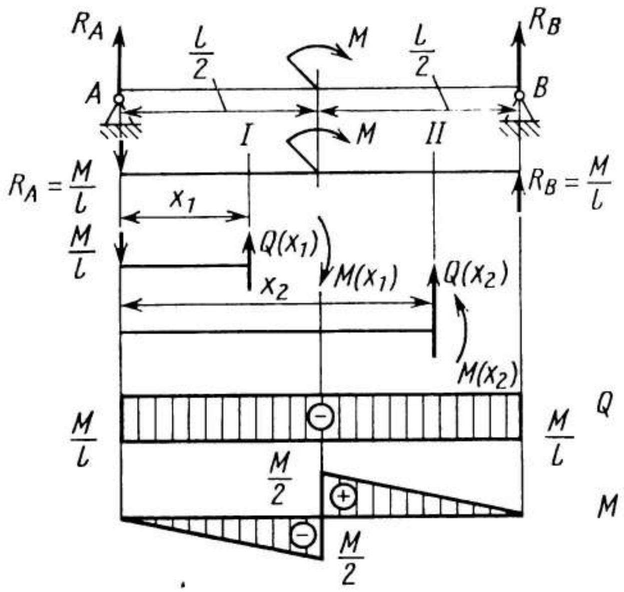


Fig. 112.
Action of a bending moment onto a simply supported beam

Solution. Determine the support reactions R_A and R_B:

$$M_A = R_B l - M = 0, \quad R_B = \frac{M}{l}$$

$$\sum F = R_A + R_B = 0, \quad R_A = -R_B = -\frac{M}{l}$$

The minus sign at the second reaction indicates that R_A should act in the direction opposite to the adopted. This circumstance will be taken into account when deriving the expressions for Q and M. As in the previous example, let us divide the beam into two portions and write the expressions for Q and M:

for portion I

$$Q(x_1) = -\frac{M}{l}$$

$$M(x_1) = -\frac{M}{l}x_1$$

where $0 \leqslant x_1 \leqslant l/2$; with $x = 0$, $M(0) = 0$ and with $x = l/2$, $M(l/2) = -M/2$;

for portion II

$$Q(x_2) = -\frac{M}{l}$$

$$M(x_2) = -\frac{M}{l}x^2 + M$$

where $l/2 \leqslant x_2 \leqslant l$; with $x_2 = l/2$, $M(l/2) = M/2$ and with $x_2 = l$, $M(l) = 0$.

As may be seen from these expressions, the lateral force is constant along the entire length of the beam and is equal to $-M/l$. The bending moment in portion I decreases from zero to $-M/2$ and in portion II, decreases from $M/2$ to zero. Let us now construct the diagrams of Q and M (see Fig. 112). As may be seen in them, two jumps are observed at the beginning and end of the Q diagram, which are equal to the support reactions R_A and R_B.

Where the moment is applied, the bending moment diagram has a jump equal in magnitude to the active moment M.

Example 3. In conclusion, let us consider the effect of a uniformly distributed load q on a simply supported beam (Fig. 113).

Solution. The support reactions are:

$$R_A = R_B = \frac{ql}{2}$$

where ql is the resultant of the distributed load, applied in the mid of the beam (q is measured in newtons per metre).

The expressions for Q and M will be:

$$Q(x) = R_A - qx$$

$$M(x) = R_A x - \frac{qx^2}{2} = \frac{q}{2}(lx - x^2)$$

with $x = 0$, $Q(0) = R_A = ql/2$ and $M(0) = 0$

with $x = l/2$, $Q(l/2) = 0$ and $M(l/2) = ql^2/8$

with $x = l$, $Q(l) = -ql/2$ and $M(l) = 0$

The lateral force decreases linearly from $ql/2$ to $-ql/2$ and the bending moment varies along the beam

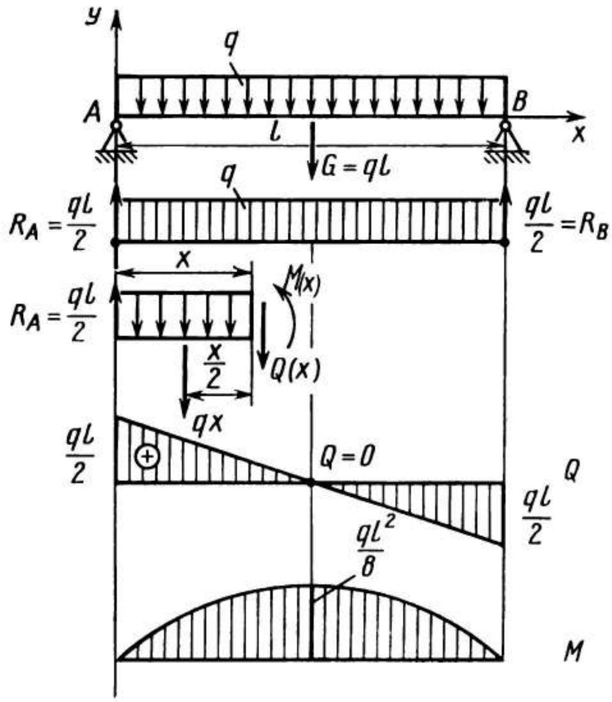


Fig. 113.
A simply supported beam with a uniformly distributed load

length non-linearly (parabolically) and attains the maximum value in the beam mid. The diagram of Q and M are shown in Fig. 113. As may be seen from the diagrams, the bending moment attains the extremal value in the point where the lateral force is equal to zero.

Let us determine the correlation between the normal bending stress and the bending moment M in a beam. Consider the case of pure bending of a beam (Fig. 114) when $Q = 0$ and only bending moment acts in a section. Experience shows that the relationship for σ in pure bending is also applicable for determining the normal stress in lateral bending.

Let us consider the positions of two cross-sectional planes I and II spaced a small distance dx from each

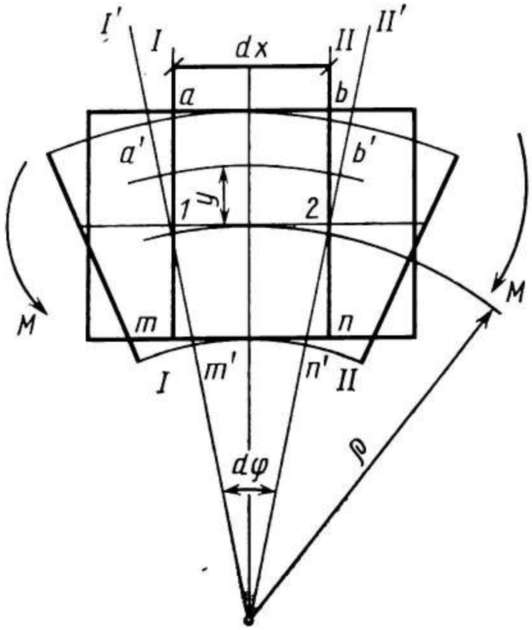


Fig. 114.
Deformation of a bar in pure bending

other (see Fig. 114). The hypothesis of plane sections holds true for bending as well as for tension and torsion. Sections I and II, which were plane before bending, have remained such after bending (I' and II'), but turned through a certain small angle $d\varphi$ and continue to be perpendicular to the external surfaces of the beam. The upper fibres of the beam became stretched, i.e. increased in length from ab to $a'b'$, whereas the lower fibres contracted from mn to $m'n'$.

By logical reasoning, there should be a parting line *1-2* between the upper and lower fibres, which does not change its length on bending and is called the neutral layer. In pure bending, this line takes the form of a circular arc of radius ρ. Let us consider the deformation of a particular fibre *cd* of the beam, which is at a distance y from the neutral layer *1-2* (Fig. 115). Let a straight line *2e* parallel to *1c* be drawn from point *2*. Then the line section *ed* will be equal to the elongation of fibre *cd*. Since the angle $d\varphi$ is small, the arcs *ed* and *1-2* can be determined with a good accuracy as $ed = y d\varphi$ and $1\text{-}2 = \rho d\varphi$, whence it follows that

$$\frac{ed}{dx} = \frac{y}{\rho}$$

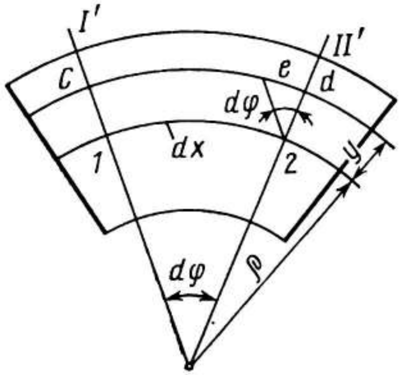


Fig. 115.
Deformation of bar fibres in bending

Noting that dx is the length of fibre *cd* before deformation (see Fig. 114), it becomes clear that ed/dx is the strain of fibre *cd*:

$$\varepsilon = \frac{y}{\rho}$$

This relationship shows that the strain of a fibre is directly proportional to the distance y of that fibre from the neutral layer. The beam experiences maximal strains in the points of a cross section which are at the largest distance from the neutral layer. The stress can be found by using Hooke's law:

$$\sigma = E\varepsilon$$

whence it follows after substitution:

$$\sigma = E\frac{y}{\rho}$$

The normal bending stress in a cross section of a beam is directly proportional to the distance from the neutral axis of the beam. Using this relationship, it is possible to construct the diagram of normal stresses in a cross

section of a beam (Fig. 116). In the neutral layer, there are neither normal stress nor strain. The line of intersection of the neutral layer with a cross section of the beam is called the neutral axis.

Let us determine the position of the neutral axis. It may be recalled that the sum of the projections of all forces in a cross section onto the x axis is equal to zero,

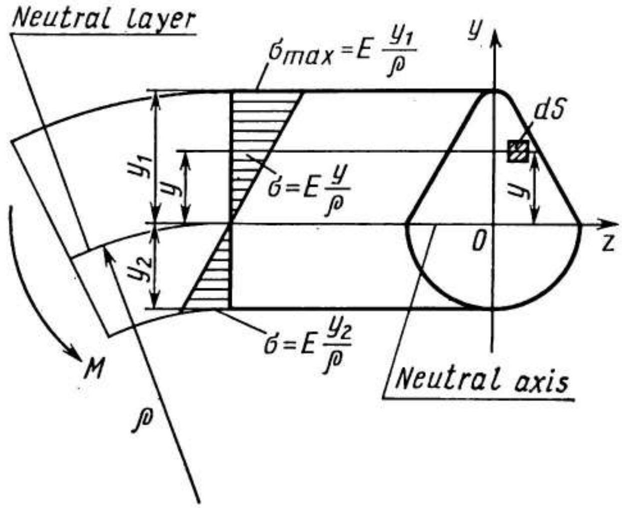


Fig. 116.
Diagram of normal bending stresses

since there are no longitudinal forces in the bending of beam.

The elementary longitudinal force acting on an elementary area dS

$$dN = \sigma\, dS = \frac{Ey}{\rho}\, dS$$

Summing over the entire area, we get:

$$N = \int_S \frac{Ey}{\rho}\, dS = 0$$

Noting that the constant E/ρ is other than zero, it follows:

$$\int_S y\, dS = 0$$

As has been demonstrated earlier in the discussion of geometrical characteristics of sections, this equality is none other than the equality to zero of the static moment of the cross-sectional area relative to the z axis: $S_z = 0$, which is an evidence that the neutral axis passes through the centre of gravity O of a section (see Fig. 116).

For numerical determination of stresses, it is essential to find the radius of curvature ρ of the neutral layer of a deformed beam. It can be obviously written that the bending moment M acting in a section is equal to the moment from normal forces. An elementary normal force acting on an elementary area dS at a distance y from the neutral axis:

$$dN = \sigma dS$$

and an elementary moment relative to the neutral axis:

$$dM = \sigma y dS$$

Summing the elementary moments over the cross-sectional area and substituting $\sigma = Ey/\rho$, we obtain:

$$M = \int_S dM = \int_S \sigma y dS = \int_S \frac{Ey^2}{\rho} dS$$

whence we find the curvature of the bent axis of beam:

$$\frac{1}{\rho} = \frac{M}{E \int_S y^2 \, dS}$$

where $\int_S y^2 \, dS = J_z$ is the moment of inertia of the cross section relative to the z axis.

Substituting the expression of curvature into the formula for σ, we finally get after simple transformations:

$$\sigma = \frac{My}{J_z}$$

Since we are interested first of all in the maximum stress, it is essential to determine from the diagram of bending moments the maximum bending moment M_{max}, after which the maximum normal stress in the section

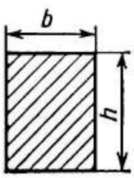
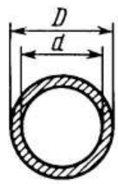
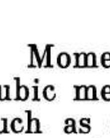
where the maximum bending moment is applied can be found by the formula:

$$\sigma_{\max} = \frac{M_{\max}}{\frac{J_z}{y_{\max}}} = \frac{M_{\max}}{W}$$

where $W = J_z/y_{\max}$ is the moment of resistance to bending.

Formulae for determining the moments of resistance to bending for the most typical shapes of section are given in Table 4 and Appendix I. These formulae are needed for practical calculations.

Table 4

Cross section	Formula of bending resistance moment W	Cross section	Formula of bending resistance moment W
	$\frac{bh^2}{6}$		$0.1D^3(1-C^4)$ $C=\frac{d}{D}$
	$0.1d^3$		

Moments of resistance to bending are measured in cubic metres. If a beam is made of a plastic material, such as steel, the strength condition is determined in terms of the maximum stress:

$$\sigma_{\max} = \frac{M_{\max}}{W} \leqslant [\sigma[$$

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For brittle materials (such as cast iron), it is required to check for strength both in tension and compression:

$$\sigma_{t\max}=\frac{M_{\max}}{\frac{J}{y_1}}\leqslant[\sigma], \quad \sigma_{c\max}=\frac{M_{\max}}{\frac{J}{y_2}}\leqslant[\sigma]_c$$

Considering the strength conditions, it is possible to solve three basic problems:

(1) problem on checking the strength: for the specified loads and geometrical dimensions of a cross section, the maximum stress in the section (called the critical section) is determined:

$$\sigma_{\max}=\left(\frac{M}{W}\right)_{\max}$$

and compared with the allowable stress $[\sigma]$;

(2) design problem: for the specified loads and allowable stresses, the cross-sectional area of a beam is determined proceeding from the moment of resistance to bending:

$$W=\frac{M_{\max}}{[\sigma]}$$

(3) problem on determining the allowable load:

$$[M] = W\,[\sigma]$$

where [M] is the allowable load determined for the critical section of a beam.

Let us consider a more complex calculation of beam bending.

Example 4. Construct the diagrams of bending moments and lateral forces for a simply supported I-shaped steel beam with overhanging ends (cantilevers) (Fig. 117). Check the strength for an I-shaped steel profile No. 18. The allowable stress $[\sigma] = 160$ MPa. The beam span is $l = 1$ m and it is loaded by a distributed load $q = 10$ kN/m.

Solution. Let the y axis be directed vertically and the x axis, horizontally. First we have to find the support

reactions R_x^A, R_y^A, and R_y^B. They can be determined from the equilibrium equations:

$$\sum F_x = R_x^A = 0$$
$$\sum F_y = R_y^A - ql + R_y^B - 2ql = 0$$
$$\sum M_A = ql^2 + ql \times 1.5l - R_y^B 3l + 2ql \times 4l = 0$$

In the last equation, the moment is determined by multiplying the force by the arm and, in the case of a

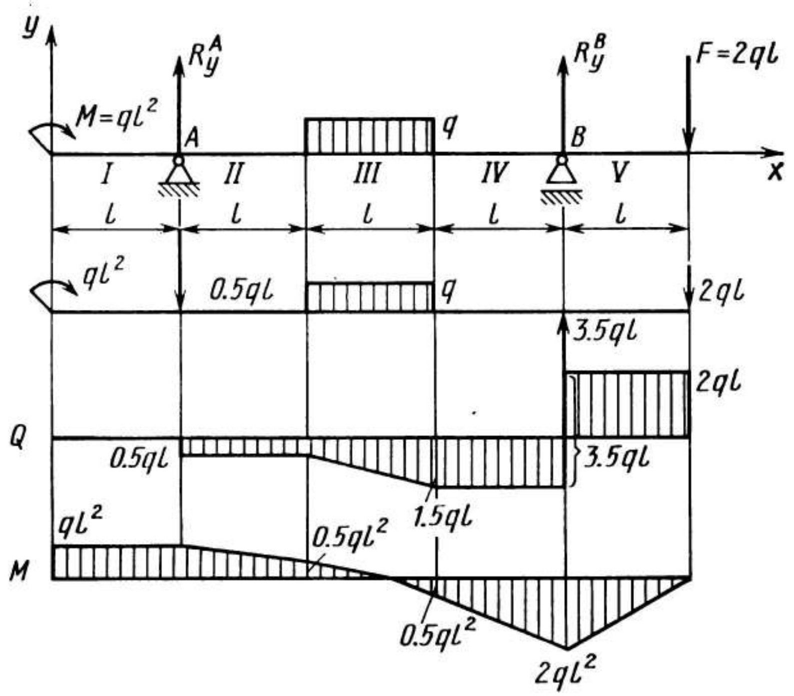


Fig. 117.
Diagram of bending moments M and shear forces Q

uniformly distributed load q, it is required to find the resultant ql and apply it in the mid of beam span. We find from the equation of moments:

$$R_y^B = \frac{10.5ql^2}{3l} = 3.5ql$$

Then the second equation gives us:

$$R_y^A = 3ql - R_y^B = 3ql - 3.5ql = -0.5ql$$

The minus sign indicates that the actual direction of force R_y^A is opposite to that adopted by us. This should be considered in the design diagram (see Fig. 117).

It is now possible to construct the diagrams. Let the beam be divided into five portions I-V in accordance with the active forces and moments. Each portion is cut by a section in order to determine by the rules which have been given earlier the internal force factors: bending moment M and lateral force Q (Fig. 118).

Portion I. Section I may be at a distance x_1 from the beam end: with $0 \leqslant x_1 \leqslant l$

$$Q(x_1) = 0$$
$$M(x_1) = ql^2$$

The moment has the plus sign, since it bends the beam so that compressed fibres are at the top of beam. The moment is constant along the entire portion.

Portion II. Section II may be at a distance x_2: with $l \leqslant x_2 \leqslant 2l$,

$$Q(x_2) = -0.5ql$$

Let us recall the rule of signs in the determination of Q (see Fig. 118). If the active force is directed downwards (when viewing the beam from left to right), then $Q < 0$; $Q(x_2)$ is constant along the whole portion:

$$Q(x_2) = -0.5ql$$
$$M(x_2) = ql^2 - 0.5ql(x_2 - l)$$
$$\text{With } x_2 = l, \quad M(l) = ql^2.$$

With $x_2 = 2l$, $M(2l) = ql^2 - 0.5ql(2l - l) = 0.5ql^2$.

Portion III. Section III may be at a distance x_3; with $2l \leqslant x_3 \leqslant 3l$,

$$Q(x_3) = 0.5ql - q(x_3 - 2l)$$
$$M(x_3) = ql^2 - 0.5ql(x_3 - l) - q\frac{(x_3 - 2l)^2}{2}$$

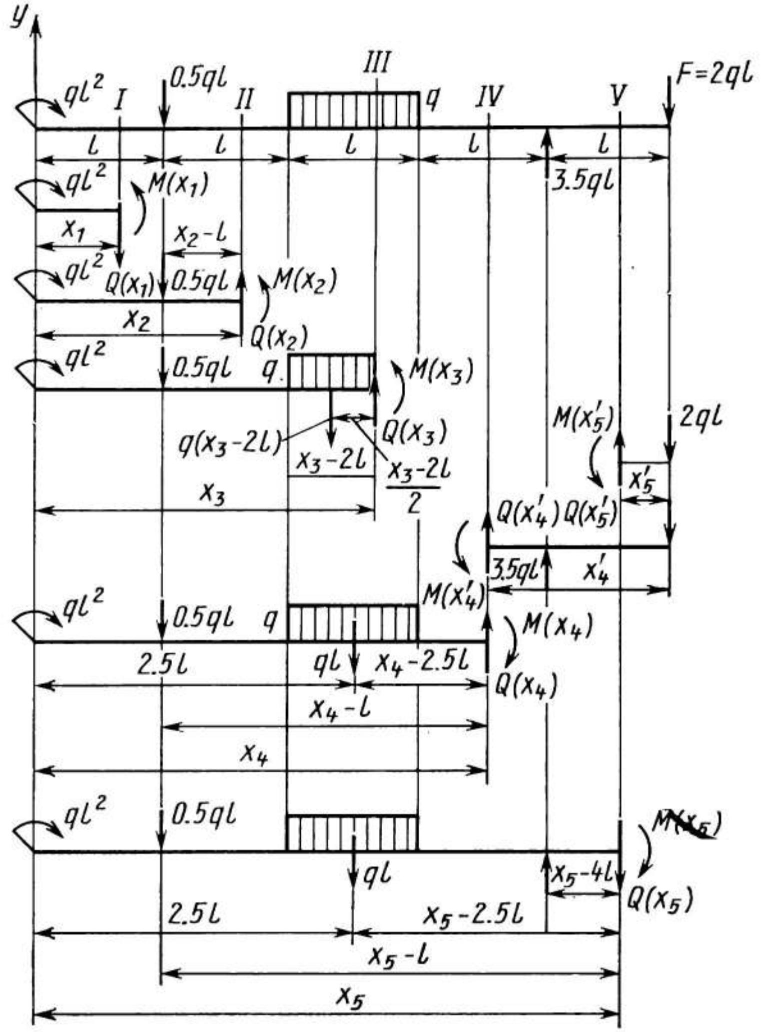


Fig. 118
Determination of M and Q in various sections of a beam

As may be seen from these equations, the lateral force $Q(x_3)$ in portion III varies linearly (x_3 to the first power) and the bending moment varies non-linearly (parabolically), i.e. proportional to the second power of x_3.

With $x_3 = 2l$,

$$Q\ (2l) = -0.5ql - q\ (2l - 2l) = -0.5ql$$

$$M\ (2l) = ql^2 - 0.5ql\ (2l - l) - q\,\frac{(2l-2l)^2}{2} = 0.5ql^2$$

With $x_3 = 3l$,

$$Q\ (3l) = -0.5ql - q\ (3l - 2l) = -1.5ql$$

$$M\ (3l) = ql^2 - 0.5ql\ (3l - l) - q\,\frac{(3l-2l)^2}{2} = -0.5ql^2$$

Then it will be easier to consider portion V and to determine section V by counting x_5' from the right-hand end of the beam. It should be considered that the rule of signs for the lateral force will then be changed (see Fig. 104) and the force will be positive if directed downwards. The rule of signs for the moment remains the same: the moment is positive if compressed fibres are at the top of beam (see Fig. 105).

Portion V. Section V may be at a distance x_5' which is counted from the right-hand end of the beam:

$$Q\ (x_5') = 2ql$$

$$M\ (x_5') = -2qlx_5'$$

With $x_5' = 0$,

$$M\ (0) = 0 \quad \text{and} \quad Q\ (0) = 2ql$$

and with $x_5' = l$,

$$M\ (l) = -2ql^2 \quad \text{and} \quad Q\ (l) = 2ql$$

These results are sufficient for constructing the diagrams of bending moments M and lateral forces Q (see Fig. 117).

The Q diagram. In the first portion, $Q\ (x_1) = 0$, because of which we begin the diagram construction with the second portion. In the point of force application $R_y^A = -0.5ql$, and therefore, there will be a downward jump in the diagram (Q is negative) by the magnitude of force, after which the force remains constant along the entire portion *II*.

In portion *III* where the distributed load is applied, we have obtained a linear variation of force Q, and therefore, the two known values of Q, at the beginning with $x_3 = 2l$ and at the end of portion with $x_3 = 3l$, can be connected by a straight line. In portion *IV*, there are no new forces, and therefore, Q is constant and equal to the initial value: $Q(x_4) = -1.5ql$.

At the beginning of portion V, where the force R_y^B is applied, there is an upward jump equal to the magnitude of force, $3.5ql$, and further along the entire portion V the force is constant: $Q(x_5) = 2ql$. At the end of portion V, where the force F is applied, there is a downward jump by $2ql$.

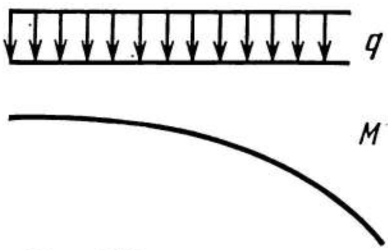


Fig. 119.
Rule for determining the convexity of M diagram under the action of load q

The M diagram. In the point of application of the moment $M = ql^2$, the M diagram has a jump equal to the magnitude of the moment. The sign of moment is determined by the position of compressed fibres: they are at the top of beam, and therefore, the moment has the plus sign and should be laid off upwards from the zero line. The moment is constant along the entire portion I and its diagram is a straight line parallel to the beam axis. In portion II, there are moments at $x_2 = l$ and $x_2 = 2l$. These moments are connected by a straight line, since the moment in this portion varies linearly depending on x_2.

In portion *III*, the uniformly distributed load q is applied, which determines that the dependence of M on x_3 becomes nonlinear. The variations of the moment $M(x_3)$ in portion *III* are described by a parabolic curve. The following rule should be remembered: the parabola of moments is always convex against the action of a distributed load q (Fig. 119). This rule can be memorized easily if the load is imagined as a 'rain' and a convex diagram, as a roof.

Knowing the initial and final value of $M(x_3)$ in por-

tion *III* and having determined by the 'rain-and-roof' rule that the curve of moment is convex upwards, it is easy to construct the diagram of $M\ (x_3)$ for portion *III*. After that, it is more expedient to consider portion *V*. At the right-hand end of the beam, $M = 0$ and at the left-hand end of portion *V*, the moment $M = -2ql^2$.

These two moments can be connected by a straight line, since $M\ (x_5')$ in portion *V* varies linearly. After that, the gap in portion *IV* can be filled in by drawing a straight line. The construction of M and Q diagrams can be checked by considering successively the portions *IV* and *V*. In practical construction of diagrams, however, the check procedure can be omitted.

Portion IV. Section *IV* may be at a distance x_4 ($3l \leqslant x_4 \leqslant 4l$)

$$Q\ (x_4) = -0.5ql - ql = -1.5ql$$

$$M\ (x_4) = ql^2 - 0.5qlx_4 + 0.5ql^2 - qlx_4 + 2.5ql^2$$
$$= 4ql^2 - 1.5qlx_4$$

With $x_4 = 3l$,

$$M\ (3l) = 4ql^2 - 4.5ql^2 = -0.5ql^2$$

and with $x_4 = 4l$,

$$M\ (4l) = 4ql^2 - 6ql^2 = -2ql^2$$

Portion V. Section *V* may be at a distance x_5 ($4l \leqslant x_5 \leqslant 5l$):

$$Q\ (x_5) = 2ql$$

$$M\ (x_5) = -10ql^2 + 2qlx_5$$

With $x_5 = 4l$,

$$M\ (4l) = -10ql^2 + 2ql \times 4l = -2ql^2$$

and with $x_5 = 5l$,

$$M\ (5l) = -10ql^2 + 10ql^2 = 0$$

As may be seen, the values of Q and M for portions *IV* and *V* coincide with those obtained earlier.

Let us now check the strength of the beam. To do this, we first find in the diagram of bending moments the

critical section of the beam, i.e. the section where the maximum bending moment is applied. This section coincides with the support B:

$$M_{\max} = 2ql^2 = 2 \times 10 \times (10^3)^2 = 2 \times 10^7 \text{ N mm}$$

where $q = 10$ kN/m $= 10$ N/mm, $l = 1$ m $= 10^3$ mm.

The maximum normal stress is found by the formula:

$$\sigma_{\max} = \frac{M_{\max}}{W} = \frac{2 \times 10^7}{143 \times 10^3} = 140 \text{ MPa}$$

where W is the moment of resistance to bending. As may be found in a reference book for I-shaped steel profile No. 18, $W = 143$ cm^3 $= 143 \times 10^3$ mm^3.

The condition of bending strength can be written as follows:

$$\sigma_{\max} \leqslant [\sigma]$$

As may be seen, the condition is satisfied:

$$\sigma_{\max} = 140 \text{ MPa} \quad \text{and} \quad [\sigma] = 160 \text{ MPs}$$

Review Questions

1. What kinds of simple loading do you know?
2. What is the internal force factor that appears in tension and compression?
3. What is the sign of a tensile force and how is it determined?
4. What is a diagram of forces?
5. How is the stress in tension and compression determined?
6. How is the strain of a bar determined?
7. What is the rigidity of a bar in tension?
8. What is the principal rule for construction of diagrams of longitudinal forces, torsional moments, lateral forces, and bending moments?
9. How are the conditions of strength in tension, compression, shear, torsion, and bending formulated?
10. What three types of problems can be solved by using the conditions of strength?
11. How is Hooke's law in shear and tension written?

12. What is the rigidity of a bar section in shear?
13. What stresses are determined in strength calculations of rivets and welded joints?
14. How is the rule of signs of torsional moments formulated?
15. How can the maximum shear stress in torsion of a round shaft be determined?
16. How is the absolute or total twisting angle determined?
17. How is the condition of rigidity of a shaft formulated?
18. Does the hypothesis of plane sections hold true in torsion of a rectangular bar?
19. What internal force factors appear in bending of a bar?
20. What is pure bending?
21. How to formulate the rule of signs for construction of bending moment diagrams?
22. What support reactions do appear in rigidly fixed and movable and immobile hinged supports?
23. How is the maximum normal stress in bending of a bar determined?
24. What is the moment of resistance to bending?

Chapter Five

Combined Loading

As distinct from simple kinds of loading, combined loading, or resistance to combined stress, is characterized by the appearance of two or more internal force factors, which may be either forces or moments, or both. All cases of combined loading are solved by using the superposition principle, i.e. the combined stresses are found by adding the stresses produced by each force factor. Combined loading includes unsymmetrical bending, eccentric tension or compression, and various combinations of simple kinds of loading.

5.1. Unsymmetrical Bending

In unsymmetrical bending, the plane in which the bending load is applied does not coincide with the main axes of inertia of the cross-sectional area of a bar. Figure 120 gives a three-dimensional view of a bar loaded by

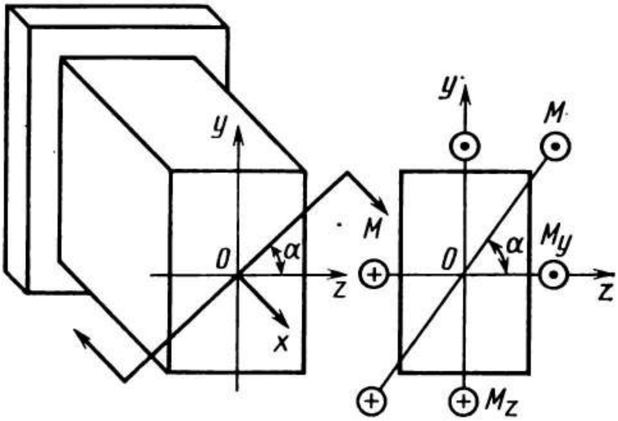


Fig. 120.
Loading scheme of a bar in unsymmetrical bending

a bending moment M (on the left) and the cross section of the bar (on the right). The axes y and z are the main central axes of the bar cross section. As may be seen, the plane of the bending moment does not coincide with the axes y and z, but makes an angle α with the z axis.

The bending moment M can be resolved into component moments relative to the axes y and z:

the bending moment relative to the y axis

$$M_y = M \cos \alpha$$

and that relative to the z axis

$$M_z = M \sin \alpha$$

Thus, each point of the bar cross section will experience stresses from two bending moments, M_y and M_z.

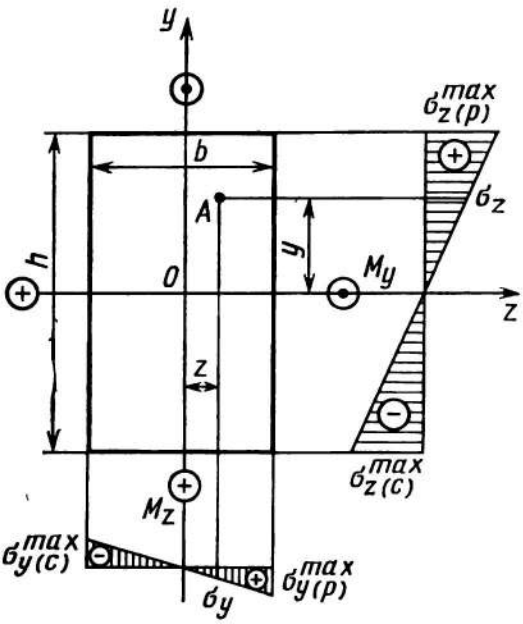


Fig. 121.
Determination of stresses in point A

In an arbitrary point A with the coordinates y and z (Fig. 121), the respective stresses will be:
from M_y

$$\sigma_y = \frac{M_y}{J_y} z$$

and from M_z

$$\sigma_z = \frac{M_z}{J_z} y$$

Adding these stresses by the superposition principle, it is possible to determine the stresses in an arbitrary point A:

$$\sigma_A = \sigma_y + \sigma_z = \frac{M_y}{J_y} z + \frac{M_z}{J_z} y$$

where

$$J_z = \frac{bh^3}{12} \quad \text{and} \quad J_y = \frac{hb^3}{12}$$

are the moments of inertia relative to the main central axes y and z.

As may be seen from Fig. 121, the maximum stresses should appear in the corner points of the section:

tensile stresses

$$\sigma_{t\max} = \sigma_{y(p)}^{\max} + \sigma_{z(p)}^{\max}$$

and compressive stresses:

$$\sigma_{c\max} = \sigma_{y(c)}^{\max} + \sigma_{z(c)}^{\max}$$

where

$$\sigma_{y(p)}^{\max} = \sigma_{y(c)}^{\max} = \frac{M_y}{W_y}; \quad W_y = \frac{hb^2}{6}$$

$$\sigma_{z(p)}^{\max} = \sigma_{z(c)}^{\max} = \frac{M_z}{W_z}; \quad W_z = \frac{bh^2}{6}$$

The formula for the maximum stresses can be written in the final form:

$$\sigma_{\max} = \frac{M_y}{W_y} + \frac{M_z}{W_z}$$

or

$$\sigma_{\max} = M\left(\frac{\cos\alpha}{W_z} + \frac{\sin\alpha}{W_z}\right)$$

where α is the angle between the plane of bending moment M and z axis.

The condition of strength for unsymmetrical bending proceeds from the obvious statement that the maximum stress must not exceed the allowable stress:

for plastic materials for which $[\sigma] = [\sigma]_c$

$$\sigma_{\max} \leqslant [\sigma]$$

and for brittle materials for which $[\sigma]_c > [\sigma]$, it is required to check separately the maximum stresses in tension and compression:

$$\sigma_{t\max} \leqslant [\sigma] \quad \text{and} \quad \sigma_{c\max} \leqslant [\sigma]_c$$

As with simple kinds of loading, the strength conditions for unsymmetrical bending make it possible to solve three kinds of problems: check problems, design

problems, and problems on determining the allowable load.

Example. Let us solve a design problem on unsymmetrical bending. A steel beam of rectangular cross section is loaded by a bending moment $M = 8$ kN m directed at an angle of 30° to the z axis. Determine the geometrical dimensions of the beam cross section if its height h is twice the width b and the allowable stress $[\sigma] = 120$ MPa.

Solution. We determine the maximum stress in unsymmetrical bending of the beam and equate it to the allowable stress:

$$\sigma_{\max} = M\left(\frac{\cos\alpha}{W_y} + \frac{\sin\alpha}{W_z}\right) = [\sigma]$$

where

$$W_z = \frac{bh^2}{6} = \frac{2}{3}b^3 \quad \text{and} \quad W_y = \frac{hb^2}{6} = \frac{b^3}{3}$$

Hence, we find upon substitution:

$$\sigma_{\max} = M\left(\frac{3\cos\alpha}{2b^3} + \frac{3\sin\alpha}{b^3}\right) = [\sigma]$$

Solving this equation for b, we find the minimum width of the beam:

$$b = \sqrt[3]{\frac{3M\,(1/2\cos\alpha + \sin\alpha)}{[\sigma]}}$$

where

$$M = 8 \text{ kN m} = 8 \times 10^6 \text{ N mm}, \quad \alpha = 30°,$$

$$\cos 30° = \frac{\sqrt{3}}{2} \quad \text{and} \quad \sin 30° = 1/2$$

Hence, we have:

$$b = \sqrt{\frac{3 \times 8 \times 10^6 \left(\frac{1}{2}\frac{\sqrt{3}}{2} + \frac{1}{2}\right)}{120}} = 57 \text{ mm}$$

5.2. Eccentric Tension and Compression

In contrast to central tension (or simply tension) which has been discussed in Ch. 3, the longitudinal tensile force in eccentric tension does not pass through the centre of gravity of the section (Fig. 122). Let the point

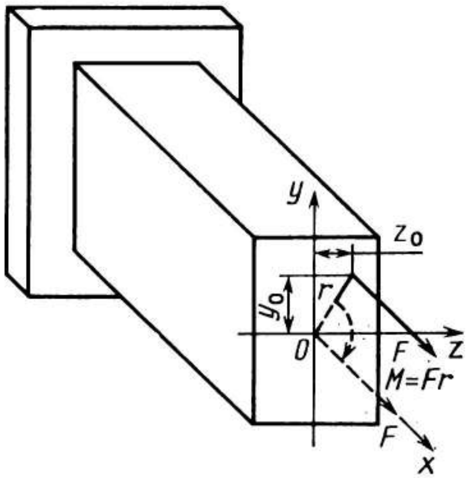


Fig. 122.
Loading scheme of a bar in eccentric tension

of application of force F be determined by the coordinates y_0 and z_0. Using the rule of force transfer given in Ch. 1, force F can be transferred to point O, the centre of gravity of the section, by adding a bending moment to it:

$$M = Fr$$

Thus, eccentric tension or compression can be reduced to the combined action of tension (or compression) and bending. As has been demonstrated in Sec. 5.1, the bending moment M creates unsymmetrical bending and can be resolved into components along the axes y and z (see Fig. 120). Thus, three internal force factors appear in each section of the bar: a longitudinal force N and two bending moments: M_y relative to the y axis and M_z relative to the z axis (Fig. 123). The actions of the force factors in the planes yOx and zOx are shown in Fig. 124.

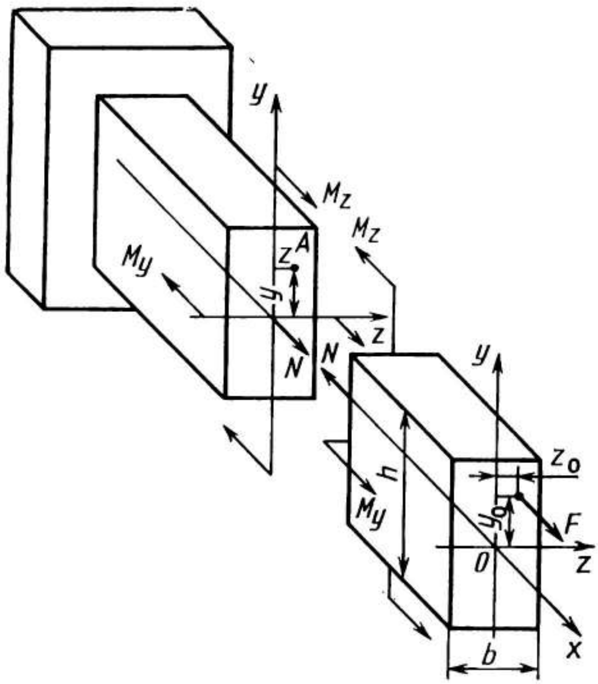


Fig. 123.
Determination of internal forces in eccentric tension of a bar

The internal force factors can be found by the method of sections. Considering the equilibrium conditions for the rejected part of the bar, it may be established (see Fig. 124) that:

$$N = F, \quad M_z = Fy_0, \quad \text{and} \quad M_y = Fz_0$$

Let us now consider how the stresses in an arbitrary point A (y, z) of the section can be determined on the basis of the superposition principle. The stress from the longitudinal force N is:

$$\sigma_N = \frac{N}{S}$$

where $S = bh$ is the cross-sectional area of the bar.

The stress from the bending moment relative to z axis, M_z, is:

$$\sigma_z = \frac{M_z}{J_z} y = \frac{Fy_0}{J_z} y$$

where $J_z = bh^3/12$ is the moment of inertia of the section relative to the z axis.

The stress from the bending moment relative to the y axis, M_y, is:

$$\sigma_y = \frac{M_y}{J_y} z = \frac{Fz_0}{J_y} z$$

where $J_y = hb^3/12$ is the moment of inertia of the section relative o the y axis.

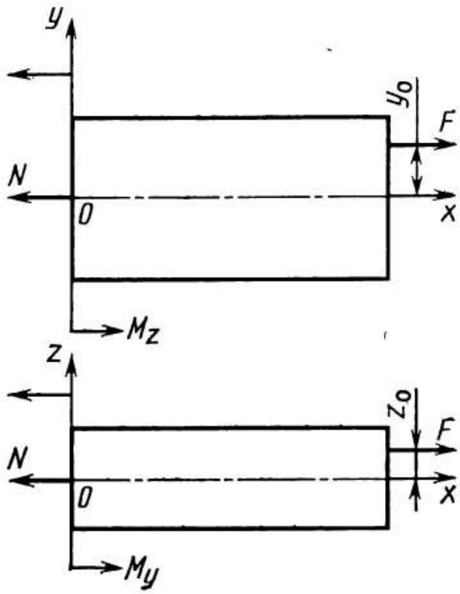


Fig. 124.
Action of force factors in planes yOx and zOx

The total (combined) stress in point A (y, z) is found by adding the stresses from all force factors: N, M_y, and M_z:

$$\sigma = \frac{N}{S} + \frac{Fy_0}{J_z} y + \frac{Fz_0}{J_y} z$$

This expression is an algebraic sum in which each term may be either positive or negative.

The sign of the first term is determined by the longitudinal force N: $N > 0$ in tension and $N < 0$ in com-

pression. The signs of the other terms are determined by the coordinates of the points of force application (y_0, z_0), the coordinates of the point for which the stress is determined (y, z), and by the sign of force F.

Example. A bar is compressed by a force F applied in point B. Find the normal stress in point A (Fig. 125).

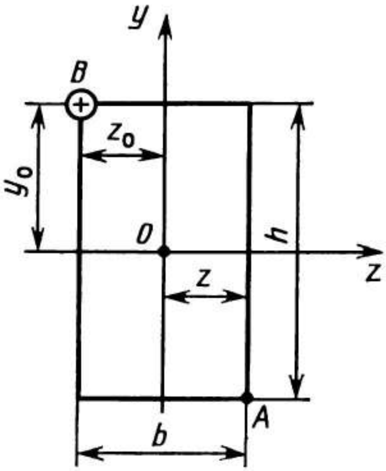


Fig. 125.
Cross section of a bar compressed by force F

Solution. The cross-sectional dimensions of the bar are: $b = 1$ cm $= 10$ mm, and $h = 2$ cm $= 20$ mm, and the force $F = 1$ kN $= 10^3$ N.

The stress in point A can be found by the formula:

$$\sigma_A = \frac{N}{S} + \frac{Fz_0}{J_y} z + \frac{Fy_0}{J_z} y$$

where $N = -F$ is the compressive force; $z_0 = -b/2$ and $y_0 = h/2$ are the coordinates of point B; $z = b/2$ and $y = -h/2$ are the coordinates of point A; $J_z = bh^3/12$, $J_y = hb^3/12$ are the moments of inertia; and S is the cross-sectional area.

Substituting for N, F, z, y, z_0, y_0, J_y, and J_z into the formula given above, we obtain:

$$\sigma_A = \frac{-F}{bh} + \frac{(-F)(-b/2)}{hb^3/12}(b/2) + \frac{(-F)(h/2)}{bh^3/12}(-h/2)$$
$$= \frac{5F}{bh} = \frac{5 \times 10^3}{10 \times 20} = 25 \text{ MPa}$$

Thus, as may be seen from this example, the eccentric compressive force F applied in point B has produced tensile normal stresses.

Strength calculations in eccentric tension and compression are similar to those in unsymmetrical bending.

5.3. Combined Bending and Torsion of a Round Bar

In the general case of combined loading consisting of a number of simple loadings, it is possible, as has been demonstrated in Secs. 5.1 and 5.2, to apply the principle of superposition (for moments and forces) and to add the stresses produced by each kind of loading.

In the case of combined action of bending and torsion, it is possible to determine separately the stresses produced

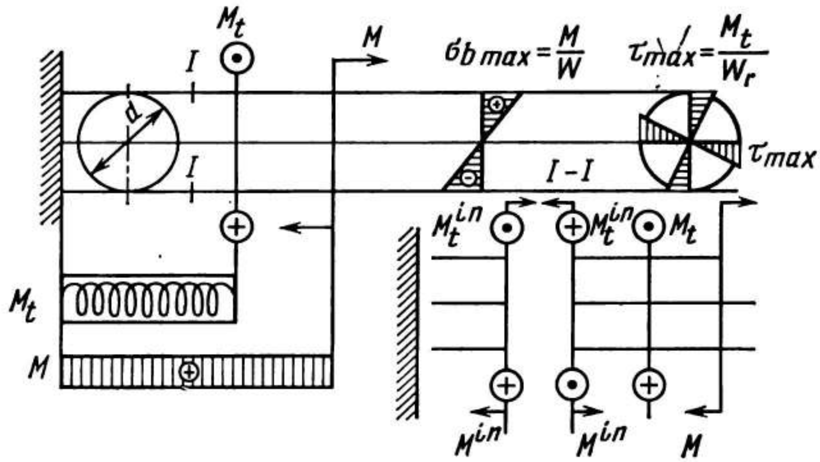


Fig. 126.
Combined action of torsion and bending moment in a beam

by each of the moments Let us consider a cantilever beam of a round cross section loaded by a torsional moment M_t and bending moment M (Fig. 126).

The distribution of moments and associated stresses along the length of the beam is shown in the same figure. In an arbitrary section I there act two internal force factors: M_t^{in} and M^{in} which, following from the equilibrium of rejected part of the beam, is the sum of all active moments:

$$M_t^{in} = \sum M_t$$
$$M^{in} = \sum M$$

In the further discussion, the superscript 'in' will be omitted.

The stresses produced by each of the moments can be found by the formulae given earlier. Let us give here the expressions for the maximum bending stresses. The normal stresses:

$$\sigma_{\max}=\frac{M}{W}$$

where $W = 0.1d^3$ is the moment of resistance to bending, and the torsional stresses:

$$\tau_{\max}=\frac{M_t}{W_r}$$

where $W_r = 0.2d^3$ is the moment of resistance to torsion.

It should be emphasized that torsional moments and tangential stresses act in the plane of a section, whereas

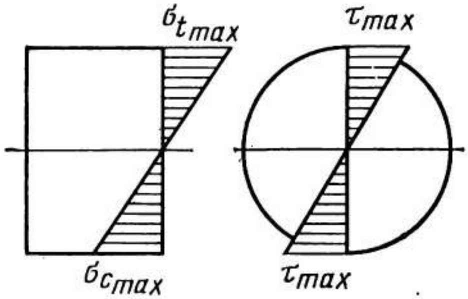


Fig. 127.
Diagrams of stresses σ and τ

bending moments and normal stresses act in a plane perpendicular to a section (Fig. 127). In this figure, $\sigma_{t\max}$ is the maximum tensile stress and $\sigma_{c\max}$ is the maximum compressive stress.

Example. Find the maximum stresses from bending and torsion of a round bar built-in at one end. The bar diameter is 10 cm, the length is 0.25 m. The bar is acted upon by a bending force of 4 kN, bending moment $M = 2$ kN m, and torsional moments $M_{t_1} = 1$ kN m and $M_{t_2} = 3$ kN m.

Solution. Let us first construct the diagrams of bending and torsional moments (Fig. 128). The bar is divided into four portions along its length. Using the method of

sections, we consider the sections in each portion of the bar and determine the internal force factors M_t and M.

Portion I. In section I, only the force F produces the bending moment

$$M(x_1) = -Fx_1$$

where the minus sign indicates that the beam is bent by the force F so that compressed fibres are at the bottom.

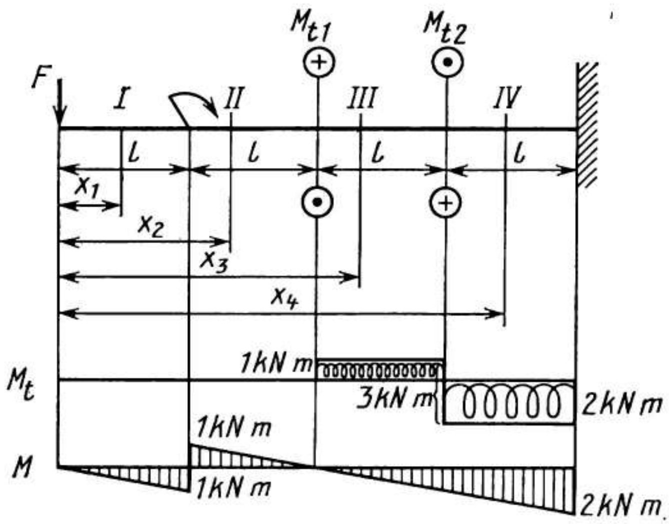


Fig. 128.
Calculation of a beam in combined loading

With $x_1 = 0$ and $x_1 = l$, it may be written:

$M(0) = 0$

$M(l) = -Fl = -4 \times 0.25 = -1$ kN m

Portion II. We take an arbitrary section II in this portion, at a distance x_2 from the end of the beam. The bending moment is produced by the force F and moment M:

$$M(x_2) = -Fx_2 + M$$

where $l \leqslant x_2 \leqslant 2l$.

With $x_2 = l$, $M(l) = -Fl + M = -1 + 2 = 1$ kN m; with $x = 2l$, $M(2l) = -2Fl + M = -2 + 2 = 0$.

Portion III. In an arbitrary section *III* in this portion, the force F and moment M create a bending moment, while the torque M_{t1} creates a torsional moment $(2l \leqslant x_3 \leqslant 3l)$;

$$M(x_3) = -Fx_3 + M;\ M(2l) = 0;\ M(3l) = -1 \text{ kN m};$$
$$M_t(x_3) = M_{t1} = 1 \text{ kN m}$$

Portion IV $(3l \leqslant x_4 \leqslant 4l)$:

$$M(x_4) = -Fx_4 + M;\quad M(3l) = -1 \text{ kN m};$$
$$M(4l) = -2 \text{ kN m};$$
$$M_t(x_4) = M_{t1} - M_{t2} = 1 - 3 = -2 \text{ kN m}$$

These results are sufficient for constructing the diagrams of bending and torsional moments (see Fig. 128). The diagrams reveal clearly the most heavily stressed point of the beam, its built-in end with the maximum bending and torsional moments equal to 2 kN m.

Now the stresses can be determined. The maximum normal stress appears due to the maximum bending moment:

$$\sigma_{\max} = \frac{M_{\max}}{W} = \frac{2 \times 10^6}{10^5} = 20 \text{ MPa}$$

where

$$M_{\max} = 2 \text{ kN m} = 2 \times 10^6 \text{ N mm}$$
$$W = 0.1d^3 = 0.1\,(10^2)^3 = 10^5 \text{ mm}^3$$
$$d = 10 \text{ cm} = 10^2 \text{ mm}$$

The maximum tangential stress is determined by the maximum torsional moment:

$$\tau_{\max} = \frac{M_{t\max}}{W_r} = \frac{2 \times 10^6}{0.2 \times 10^6} = 10 \text{ MPa}$$
$$M_{t\max} = 2 \text{ kN m} = 2 \times 10^6 \text{ N mm};$$
$$W_r = 0.2d^3 = 0.2 \times 10^6 \text{ mm}^3$$

5.4. Combined Shear and Torsion

An example of combined shear and torsion is the stressed state of a helical spring. The internal force

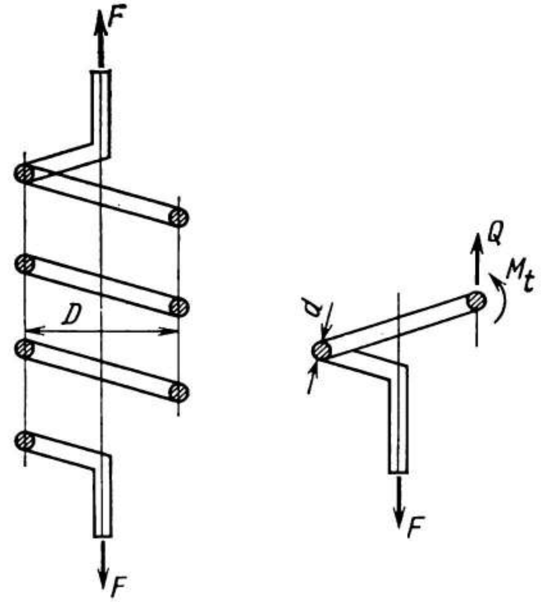


Fig. 129.
Calculation of a spring

factors Q and M_t for the case considered can be determined by the method of sections:

$$Q = F; \qquad M_t = \frac{FD}{2}$$

The tangential stresses appearing in the spring are determined separately for each force factor:

from the shear force Q

$$\tau_Q = \frac{Q}{S} = \frac{F}{S}$$

where $S = \pi d^2/4$ is the cross-sectional area of spring wire;

from the torsional moment M_t

$$\tau_M^{\max} = \frac{M_t}{W_r} = \frac{FD}{2W_r}$$

where $W_r = \pi d^3/16$ is the moment of resistance to torsion.
The total maximum stress in the spring sections

$$\tau_{\max} = \tau_Q + \tau_M^{\max} = \frac{4F}{\pi d^2} + \frac{8FD}{\pi d^3} = \frac{8FD}{\pi d^3}\left(1 + \frac{d}{2D}\right)$$

The term $d/2D$ is much less than unity and can be neglected. In that case the stresses due to shearing will be disregarded, since they are substantially lower than those due to torsion.

Example. Determine the diameter d of steel wire to make a helical spring of the outside diameter $D = 0.1$ m which will be tensioned by a force $F = 1$ kN (Fig. 129). The allowable stress of steel wire $[\tau] = 200$ MPa.

Solution. Using the obvious strength condition $\tau_{\max} \leqslant [\tau]$ and neglecting the term $d/2D$, we solve the formula given above for d and find:

$$d = \sqrt[3]{\frac{8FD}{\pi[\tau]}} = \sqrt[3]{\frac{8 \times 1 \times 10^3 \times 100}{\pi \times 200}} = 11 \text{ mm}$$

5.5 Determination of Stresses in Vessels and Pipelines

Various reservoirs are mostly made as spherical or cylindrical shells. The thickness of walls of a reservoir is usually small compared with the diameter, because of which they are commonly called thin-walled vessels.

Let us derive the formulae for determining the stresses that appear in the walls of reservoirs under the action of internal pressure. This can be done by using the method of sections, a universal method in strength of materials. Let us first consider a spherical reservoir of a radius R and wall thickness δ under the action of an internal pressure p (Fig. 130).

It should be clear from the symmetry of a spherical reservoir that the stresses developed in it are the same along both the x and y axis (Fig. 131).

From the conditions of equilibrium for the rejected portions of the reservoir it may be established that

$$\sigma_x \times 2\pi R\delta = \sigma_y \times 2\pi R\delta = p\pi R^2$$

where $2\pi R\delta$ is the cross-sectional area of the reservoir shell.

Hence,

$$\sigma_x = \sigma_y = pR/2\delta$$

Thus, the stresses in a spherical reservoir are directly proportional to the pressure and the radius of the sphere, and inversely proportional to the wall thickness.

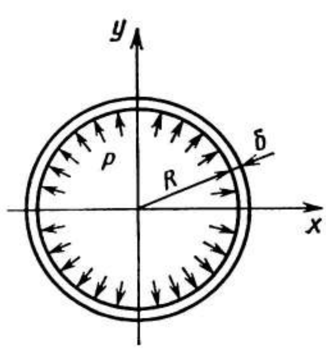


Fig. 130.
Cross section of a vessel under pressure

Let us now consider a cylindrical reservoir of a radius R and wall thickness δ with a gauge pressure p acting inside (Fig. 132). A cylindrical reservoir consists of two bottoms, usually spherical, and a cylindrical shell in the mid portion.

The formulae for the stresses in the spherical bottoms have been derived above. Determine the stresses in the cylindrical portion. To do this, let us cut the cylinder by two sections: a longitudinal and a transverse. The stresses in the circumferential direction

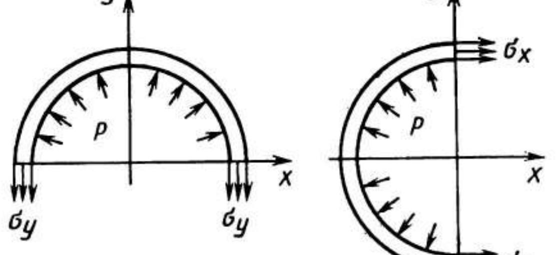


Fig. 131.
Determination of stresses in walls of a vessel

of the shell are called circumferential, or hoop, stresses and those in the longitudinal direction, longitudinal,

or axial, stresses. Let a portion of length l be cut from the cylindrical portion (Fig. 133*a*). Projecting all forces onto the y axis and equating them to zero:

$$-2\sigma_y l\delta + 2pRl = 0$$

we can find the circumferential (hoop) stress:

$$\sigma_y = \frac{pR}{\delta}$$

The axial stress σ_x can be found from the equilibrium condition for the rejected portion of the reservoir

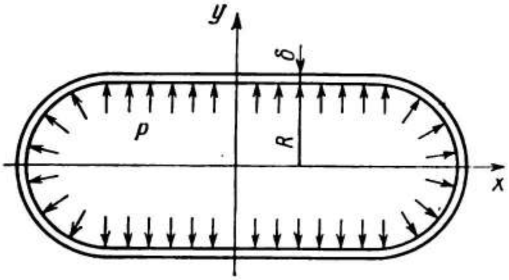


Fig. 132.
Longitudinal section of a vessel

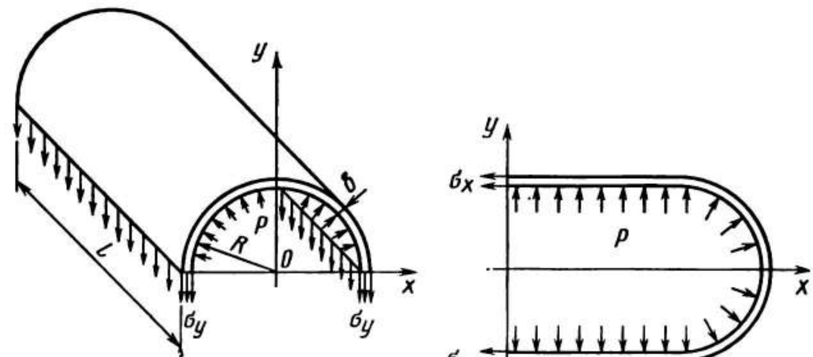


Fig. 133.
Determination of (*a*) circumferential and (*b*) axial stresses in walls of a cylinder

(Fig. 133*b*). Projecting onto the x axis the force due to stresses $\sigma_x \times 2\pi R\delta$ and that due to the internal pressure

$p\pi R^2$ and equating them to zero:

$$-\sigma_x 2\pi R\delta + p\pi R^2 = 0$$

we can find:

$$\sigma_x = \frac{pR}{2\delta}$$

Thus, the maximum stresses in a cylindrical reservoir appear in the circumferential direction (hoop stresses) and they are twice as large as the axial stresses. With the same radius and wall thickness of a cylindrical and a spherical reservoir, the maximum stresses in the former are twice those in the latter. In pipelines, the maximum stresses appear in the circumferential direction (hoop stresses). They are found, as the hoop stresses in cylindrical reservoirs, by the formula:

$$\sigma = \frac{pR}{\delta}$$

5.6. Strength Calculations in Combined Loading

In simple kinds of loading which have been discussed earlier, the strength condition consists in that the maximum stresses must not exceed the allowable stress for the given material.

Some kinds of combined loading, such as unsymmetrical bending or eccentric tension or compression, are reduced to a maximum tensile or compressive stress which must obey the strength condition:

$$\sigma_{max} \leqslant [\sigma]$$

How is it possible to estimate the strength in other cases of combined loading, for instance, in combined bending and torsion where both normal and tangential stresses are developed, or in the case of thin-walled vessels? The sole conclusion suggests itself: a combined stressed state should be reduced to simple tension.

This problem is solved by using the hypotheses or theories of strength. The tensile stress to which a combined stressed state is reduced by means of the theory of strength is called the equivalent stress σ_{eq}.

For understanding the theories of strength, it is essential to have knowledge on the stresses or, more particularly, on the stressed state in a point of a body. As has been demonstrated earlier in the book, any point can in the general case be confined within a cube so that, if the sides of the cube are diminished continuously, this will in the limit degenerate into a point.

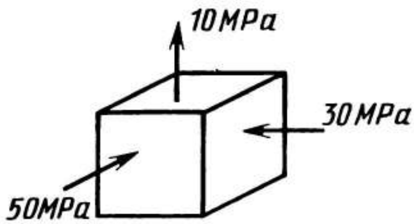


Fig. 134.
Three-dimensional stress in a point

In the general case, the total stresses p appearing in the faces of such a cube can be resolved into normal and tangential stresses. Depending on the position of a cube in space, the total stresses, and therefore, their components, can vary continuously. For instance, taking an arbitrarily oriented cube in a tensioned bar, the faces of the cube will be inclined to the line of longitudinal force and, therefore, will experience tangential stresses. It is, however, possible to find a position for the cube in which there will be no tangential stresses in its faces.

Such faces are called the main planes and the normal stresses acting in them are called the main stresses and designated σ_1, σ_2, σ_3 in the decreasing order of their magnitude, i.e. $\sigma_1 > \sigma_2 > \sigma_3$.

For instance, as shown in Fig. 134, the stressed state in a point of a body can be characterized by the following main stresses:

$$\sigma_1 = 10 \text{ MPa}, \quad \sigma_2 = -30 \text{ MPa}, \quad \text{and} \quad \sigma_3 = -50 \text{ MPa}$$

The stresses σ_2 and σ_3 are directed towards the main planes of the cube, i.e. they are compressive and accordingly have the minus sign.

We can now discuss the theories of strength by means of which the main stresses are reduced to a single tensile stress, the equivalent stress σ_{eq}.

There are several theories of strength.

The first theory, or the theory of maximum normal stresses, assumes that the equivalent stress is equal to the

maximum normal stress:

$$\sigma_{eq} = \sigma_1$$

The second theory, or the theory of maximum relative strain, determines the equivalent stress from the equality of the maximum relative strains in combined and reduced stressed states.

$$\varepsilon_{eq} = \varepsilon_{\max}$$

These two theories have been suggested long ago and have almost no practical application, since their results do not agree well with experimental data.

The third theory of strength, or the theory of maximum tangential stresses, is employed quite widely in practical calculations. According to it, the equivalent stressed state has a tangential stress equal to the maximum tangential stress of a combined stressed state:

$$\tau_{eq} = \tau_{\max}$$

In terms of normal stresses, this theory is expressed as follows:

$$\sigma_{eq} = \sigma_1 - \sigma_3$$

where σ_1 and σ_3 are respectively the maximum and the minimum main stresses. For brittle materials, the third theory is generalized by introducing a factor $\nu = [\sigma]/[\sigma]_c$:

$$\sigma_{eq} = \sigma_1 - \nu\sigma_3$$

For plastic materials, as is known, $[\sigma] = [\sigma]_c$, i.e. $\nu = 1$, and the generalized theory gives the particular result as above: $\sigma_{eq} = \sigma_1 - \sigma_3$.

In cases when bending and torsional moments are acting on a bar, and therefore, the stresses σ and τ appear in the cube faces, the third theory of strength first determines the main stresses σ_1 and σ_3, after which the equivalent stress σ_{eq} can be found by the following relationship:

$$\sigma_{eq} = \sqrt{\sigma^2 + 4\tau^2}$$

This relationship can be used for estimating the strength of round bars subjected to combined action of bending and torsion.

The fourth theory of strength is the energetic theory. It is based on the assumption that the relative strain energy is the same for an equivalent stressed state and that being considered. Like the third theory, it has found wide application in calculations of structures.

By analogy with the simple kinds of loading discussed earlier, the general condition of strength can be written as follows:

$$\sigma_{eq} \leqslant [\sigma]$$

It should be noted that the allowable stress $[\sigma]$ is determined by considering the specified safety factor $[n]$:

$$[\sigma] = \frac{\sigma_{ult}}{[n]}$$

where σ_{ult} is the ultimate stress of the material.

Safety factors are specified strictly depending on the purpose of a structure and its dimensions, the properties and homogeneity of the material, the design diagram chosen, and on how accurately the stresses have been determined. The ultimate stress for plastic materials, such as structural carbon steels, is taken equal to their yield strength and that for brittle materials, equal to their ultimate strength.

5.7. Stability of Bars. Temperature Stresses and Deformations

Up to this point, we have considered the compression of relatively short and thick bars possessing an ample stability. Practical calculations, however, often have to deal

Fig. 135.
Scheme of loading of a rule

with bars of an appreciable length and small cross-sectional area, and therefore, of a low rigidity. An example is a measuring metal rule which bends into an arc on the application of relatively low forces to its ends (Fig. 135). The phenomenon is called instability, or loss of stability.

The force that causes a bar to loose stability is called the critical force F_{cr}. The allowable force $[F]$ is determined by introducing a safety factor of stability K_{st} by the formula:

$$[F]=\frac{F_{cr}}{K_{st}}$$

The allowable force of compression of bars has been discussed earlier and is determined by the following formula:

$$[F] = [\sigma]_c S$$

where $[\sigma]_c$ is the allowable compressive stress for a bar, and S is the cross-sectional area.

The allowable force in stability calculations of bars is determined by the similar formula:

$$[F] = \varphi [\sigma]_c S$$

where φ is the coefficient of reduction of the allowable compressive stress.

With the safety factor $K_{st} = 1.8$, the coefficient φ can be taken from Table 5, depending on the bar material and flexibility λ:

$$\lambda=\mu l \sqrt{\frac{S}{J_{min}}}$$

where μ is the coefficient of length reduction, and l is the length of the bar. With a different value of safety factor K_{st}, the coefficient φ can be found by the following relationship:

$$\varphi=\varphi_t \frac{1.8}{K_{st}}$$

where φ_t is the coefficient φ from Table 5; and J_{min} is the minimum moment of inertia of the bar cross section.

The coefficient of length reduction μ depends on the kind of fixation of bar ends. Its values for various kinds of fixation (anchorage) are indicated in Fig. 136.

Example. A bar of a length $l = 0.25$ m and rectangular cross section with the width $b = 10$ cm and height

Table 5

Flexibility λ	Coefficient of reduction of allowable stress, φ				
	Steel grades 1, 2, 3, 4	Steel grade 5	Alloy steels ($\sigma_b \geqslant 320$ MPa)	Cast iron	Wood
0	1.00	1.00	1.00	1.00	1.00
10	0.99	0.98	0.97	0.97	0.99
20	0.96	0.95	0.95	0.91	0.97
30	0.94	0.92	0.91	0.81	0.93
40	0.92	0.89	0.69	0.69	0.87
50	0.89	0.86	0.83	0.57	0.80
60	0.86	0.82	0.79	0.44	0.71
70	0.81	0.76	0.72	0.34	0.60
80	0.75	0.70	0.65	0.26	0.48
90	0.69	0.62	0.55	0.20	0.38
100	0.60	0.51	0.43	0.16	0.31
110	0.52	0.43	0.35	—	0.25
120	0.45	0.37	0.30	—	0.22
130	0.40	0.33	0.26	—	0.18
140	0.36	0.29	0.23	—	0.16
150	0.32	0.26	0.21	—	0.14
160	0.29	0.24	0.19	—	0.12
170	0.26	0.21	0.17	—	0.11
180	0.23	0.19	0.15	—	0.10
190	0.21	0.17	0.14	—	0.09
200	0.19	0.16	0.13	—	0.08

$h = 1$ cm is rigidly fixed at one end and loaded by a compressive force F at the other. Determine the allowable force F_{st} to ensure the bar stability and compare it with the allowable force in compression, $[F]_c$. The bar is made of steel with the allowable compressive stress $[\sigma]_c = 160$ MPa. The loading scheme of the bar is shown in Fig. 137.

Solution. The allowable force in compression

$$[F]_c = [\sigma]_c S = 160 \times 10^3 = 160 \times 10^3 \text{ N} = 160 \text{ kN}$$

where $S = bh = 100 \times 10 = 10^3$ mm^2.

The allowable force to ensure the bar stability

$$[F]_{st} = \varphi\,[\sigma]_c\,S = \varphi \times 160 \times 10^3 = \varphi \times 160 \text{ kN}$$

The coefficient φ is taken from Table 5 by considering the bar flexibility:

$$\lambda = \mu l \sqrt{\frac{S}{J_{\min}}}$$

where $\mu = 2$ is the coefficient of length reduction for a bar with a built-in end; $l = 0.25$ m $= 250$ mm; and $S = bh = 10^3$ mm^2.

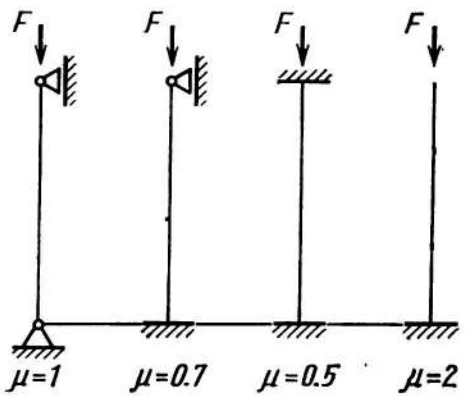


Fig. 136.
Coefficient of length reduction μ depending on the kind of end anchorage

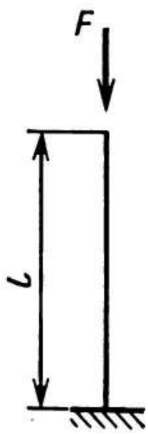


Fig. 137.
Scheme of loading of a bar

The minimum moment of inertia of the bar cross-section is:

$$J_{\min} = \frac{bh^3}{12} = \frac{100 \times 10^3}{12} = \frac{10^5}{12} \text{ mm}^4$$

Substituting the numerical values, we get:

$$\lambda = 2 \times 250 \sqrt{\frac{10^3 \times 12}{10^5}} = 175$$

The coefficient φ can be found in Table 5 by interpolation:

with $\lambda_1 = 170$, $\quad \varphi_1 = 0.26$
with $\lambda_2 = 180$, $\quad \varphi_2 = 0.23$

From the proportion

$$\frac{\lambda_2 - \lambda_1}{\lambda - \lambda_1} = \frac{\varphi_1 - \varphi_2}{\Delta\varphi}$$

we find the difference

$$\lambda_2 - \lambda_1 = 180 - 170 = 10$$

which correspondingly gives:

$$\varphi_1 - \varphi_2 = 0.26 - 0.23 = 0.03$$

Thus, with $\lambda = 175$, the coefficient $\varphi = \varphi_2 + \Delta\varphi = 0.23 + 0.015 = 0.245$

Finally, the allowable force to ensure stability

$$[F]_{st} = 0.245 \times 160 = 39 \text{ kN}$$

Thus, the force that ensures the stability of the bar in the example considered is roughly only 1/4 of the allowable force in simple compression.

As is known from the course on physics, all bodies change their dimensions on heating or cooling. Let a specimen of a length l be fastened at one end and heated from the temperature t_1 to t_2 (Fig. 138).

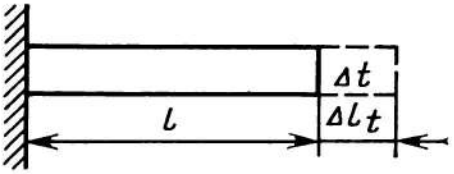


Fig. 138. Temperature deformation of a bar

The temperature change of the specimen can be determined as:

$$\Delta t = t_2 - t_1$$

The length of the specimen has increased on heating by Δl_t which is called the absolute temperature elongation:

$$\Delta l_t = \alpha \Delta t l$$

where α is the coefficient of linear expansion.

In accordance with the earlier definition, the temperature deformation can be determined as:

$$\varepsilon_t = \frac{\Delta l_t}{l} = \alpha \Delta t$$

As may be seen from this formula, the temperature deformation is proportional to the temperature change and the coefficient of linear expansion. The latter is a physical characteristic of a material. Its values for selected materials are as follows:

Cast iron	1×10^{-5}	Copper	1.7×10^{-5}
Steel	1.2×10^{-5}	Aluminium	2.2×10^{-5}

The sign of absolute temperature elongation and relative temperature deformation determines the sign of Δt: $\Delta t > 0$ on heating and $\Delta t < 0$ on cooling.

During heating or cooling of a specimen which is free at least at one end, the length of the specimen varies freely, and therefore, no stresses appear in the material. If, however, the specimen is not allowed to expand or contract freely, say, by fixing its free end, there will appear internal stresses σ_t which are called temperature stresses.

If a specimen is fastened at both ends, compressive stresses will develop in it on heating ($\Delta t > 0$) and tensile ones, on cooling ($\Delta t < 0$).

The deformation caused by temperature stresses can be determined on the basis of Hooke's law $\sigma_t = E\varepsilon_{\sigma_t}$:

$$\varepsilon_{\sigma_t} = \frac{\sigma_t}{E}$$

Since the length of a specimen fastened at both ends cannot change on a change of temperature by Δt, the sum of relative deformations ε_t and ε_{σ_t} should be equal to zero:

$$\varepsilon_t + \varepsilon_{\sigma_t} = 0$$

whence, substituting for ε_t and ε_{σ_t}, we obtain:

$$\sigma_t = -\alpha \Delta t E$$

Example. Find the temperature stresses in tramway rails on heating by 20 degrees C.

Solution. The temperature stresses are determined by the formula:

$$\sigma_t = -\alpha \Delta t E = -1.2 \times 10^{-5} \times 20 \times 2 \times 10^5$$
$$= -48 \text{ MPa}$$

where $\alpha = 1.2 \times 10^{-5}$ 1/deg and $E = 2 \times 10^5$ MPa (see Table 2).

Review Questions

1. What kinds of loading are called combined?
2. What internal force factors will appear in unsymmetrical bending?
3. What internal force factors will appear in eccentric compression?
4. What internal force factors will appear in combined bending and torsion of a round bar; in combined shear and torsion?
5. What stresses will appear in spherical and cylindrical reservoirs? Where are the maximum stresses developed?
6. What is the practical use of the theories of strength? What principal theories of strength do you know?
7. What is the safety factor?
8. What is the critical force?
9. Can the allowable force determined by the stability calculation of a bar be greater than the force calculated for the bar from the strength condition in compression?
10. What is the flexibility of bars?
11. What is the coefficient of reduction of bar length? What does it depend on?

Chapter Six

Experimental Measurement of Stress and Strain

6.1. Kinds of Tests

Experimental strength tests of materials are mainly carried out for the following purposes:

(a) to determine the actual stressed-strained state;

(b) to establish the actual magnitude of active maximum loads; and

(c) to check whether the calculations by theoretical methods have been correct and to assign safety factors.

As is known from the theory of strength of materials, there are simple and combined kinds of loading of structures. A simple loading is that when a structure or specimen is loaded by only one kind of load, say, tensile or compressive, and a combined loading implies that two or more different kinds of load, for instance, a compressive load and a torsional moment, act on the structure.

It is known from the foregoing that each kind of loading produces a particular kind of strain in the tested specimen.

Specimens and structures can be tested by destructive and non-destructive methods. In destructive tests, a specimen is loaded up to its fracture, while in non-destructive tests, loading is carried out to a certain specified level, after which the stressed state of the specimen or structure is determined and the stress-strain relationship is established.

The main characteristics and properties of structural materials are mainly determined by destructive methods of tests. They are employed in the design of new structures and for checking the operating ability of materials in seriesly manufactured structures.

Destructive tests are classified by the kind of loading: tests in tension, compression, bending, torsion, shear, impact tests, fracture tests, creep tests, endurance (fatigue) tests, high-temperature strength tests, wear an abrasion tests.

Depending on the kind or pattern of loading, it is distinguished between static, dynamic, and special tests of materials and structures.

In static tests, a continuously or stepwise increasing load is applied to a specimen up to its fracture. Examples of static loading are tests of specimens in tension, compression or torsion, pressure tests of vessels, and tests for static stability of structures.

Dynamic tests are meant as tests of specimens to fracture with the total load being applied in a very short interval of time or almost instantaneously. The time of loading constitutes only a fraction of a second and only the total work of deformation is measured. Examples of dynamic tests are impact tests of specimens. Examples of structures subjected to impact loads are aircraft undercarriages or metal sheets in explosion forming.

A separate group includes special tests, for instance, fatigue and creep tests. In fatigue tests, a specimen is subjected to a cyclic repeated load varying by a specified law (say, sinusoidal). The results of the test are used for constructing the diagram of state of the material depending on the number of loading cycles and for specifying the safe number of cycles for operation of a structure.

In creep tests, a specimen is loaded by the full specified load, say, tensile or compressive, and kept under the load for a long time. The results of tests are represented in the form of the diagram of time variation of strain under a constant load.

The main mechanical properties of materials in machine-building are determined by tests of specimens in a simple kind of loading: tests to fracture under a tensile load. The test procedure and the characteristics to be measured are specified by state standards. The results of these tests are used for plotting a tensile test diagram which is usually called the stress-strain diagram of a material.

The main characteristics that can be determined in tensile tests are the conditional proportional limit σ_{pr}, conditional limit of elasticity σ_{el}, conditional yield point (yield strength) σ_y, conditional tensile strength (ultimate strength) σ_t, initial cross-sectional area of specimen S_0,

relative elongation δ, relative contraction ψ, and modulus of elasticity E.

The word 'conditional' is usually omitted. In special creep and fatigue tests, there are determined additionally the elongation Δl, time t, alternate stresses $\pm\sigma$, and the number of cycles n.

The tensile test diagram (stress-strain diagram) of a mild steel specimen is shown in Fig. 139. The load F

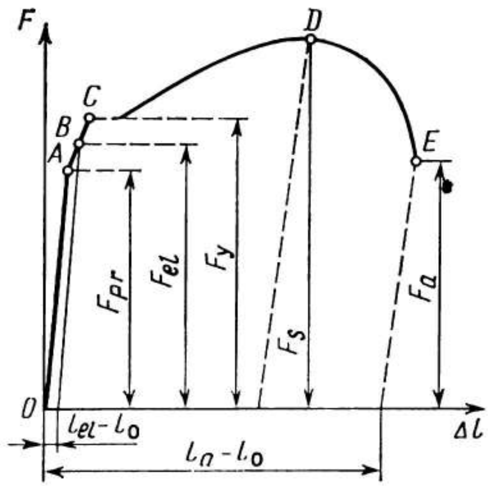


Fig. 139.
Tensile test diagram for a soft steel specimen

is laid off along the vertical axis and the specimen elongation Δl, along the horizontal. The loads in points A, B, C, D, and E are respectively the load of proportionality F_{pr}, load of elastic limit F_{el}, yield load F_y, ultimate strength load F_t, and load of actual resistance of material F_a (fracture load). As may be seen from the diagram, the relationship between load and elongation Δl in the initial portion of the curve, from point O to A, is linear. This relationship is called the law of proportionality, and the load F_{pr} at which the linear relationship changes to non-linear determines the limit of proportionality σ_{pr}.

The stress σ_{pr} is the conditional limit of proportionality above which the tangent of the $F - \Delta l$ curve in

point F_{pr} relative to the load axis increases by 50% relative to its value in the linear elastic portion.

The expression for σ_{pr} is written as follows:

$$\sigma_{pr} = \frac{F_{pr}}{S_0}$$

where S_0 is the initial cross-sectional area of the tested specimen.

For some materials, a different change of tangent is assumed, for instance 10% or 25%, and the proportionality limit then has an additional subscript, say, σ_{pr10}.

After point A, the next characteristic point is B which determines the conditional limit of elasticity σ_{el}.

The stress σ_{el} is the conditional limit of elasticity at which the residual elongation attains 0.05% of the length of specimen portion equal to the nominal length of strain gauge. The expression for σ_{el} is:

$$\sigma_{el} = \frac{F_{el}}{S_0}$$

The limit of elasticity means physically that a specimen actually returns to its original state upon removal of the load. As with σ_{pr}, a different limit can be specified for σ_{el}, say, a residual elongation of 0.005%. The actual residual elongation can be found in the diagram by drawing from point B a straight line parallel to the linear portion of the curve.

Another characteristic point in the diagram is C, after which there follows a straight section parallel to the axis of abscissae. As may be seen, in this portion of the curve the length of the specimen increases without any incre se of the load. This point defines the conditional yield poi t (yield strength) σ_y.

The stress σ_y is the conditional yield strength at which the tested specimen is deformed without any noticeable increase of the load.

The expression for σ_y is written as follows:

$$\sigma_y = \frac{F_y}{S_0}$$

The yield strength is determined by the presence of a linear horizontal portion in the tensile test diagram, which is called the 'yield step' ('yield hump'). The yield strength is determined at the residual elongation usually equal to 0.2% and is respectively designated as $\sigma_{y0.2}$. Brittle materials (glass, plastics, some alloyed steels) have no yield step in the tensile test diagram.

As may be seen from the diagram in Fig. 139, after the yield step the load increases again up to point D, and beyond that point the curve drops down more steeply than it was rising before. This point determines the ultimate rupture strength of the material in tension, σ_t.

The stress σ_t is the ultimate (limiting) strength of the material, i.e. the stress at which rupture occurs. This stress corresponds to the maximum load in the tests. The expression of the ultimate strength σ_t is as follows:

$$\sigma_t = \frac{F_t}{S_0}$$

The diagram has still another point, E, which determines the actual load F_a at which physical rupture of the specimen takes place, i.e. the latter breaks into two halves. This point determines the actual rupture stress σ_a which is expressed as the ratio of the actual load F_a at the moment of rupture to the final cross-sectional area S_f of the specimen at that moment:

$$\sigma_a = \frac{F_a}{S_f}$$

For measuring the final cross-sectional area, the two halves of a broken specimen are taken from the testing machine and put together.

The relative elongation on rupture, δ, is the ratio of the increment of specimen length to the initial length:

$$\delta = \frac{l_r - l_0}{l_0} 100\%$$

where l_r is the length of the specimen on rupture, and l_0 is its initial length before the load application.

As a specimen is stretched in test, its width and thickness naturally diminish which is characterized by the

relative contraction ψ. The relative contraction ψ is the ratio of the difference between the initial and final cross-sectional area in the point of rupture to the initial cross-sectional area:

$$\psi = \frac{S_0 - S_f}{S_0} 100\%$$

The initial cross-sectional area can be calculated by the following formulae:

for round specimens

$$S_0 = \frac{\pi d_0^2}{4}$$

and for specimens of rectangular cross section

$$S_0 = b_0 a_0$$

where d_0 is the initial diameter, b_0 is the initial width, and a_0 is the initial thickness.

Not all materials behave in tests like mild steel whose stress-strain diagram is shown in Fig. 139. The stress-

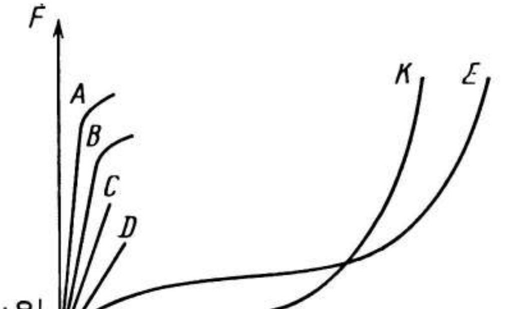


Fig. 140.
Stress-strain diagram of various materials in tension

strain diagrams for selected materials are shown in Fig. 140.

The diagrams for high-strength steel (curve A), high-strength aluminium alloy (curve B), bone (curve C), and wood (curve D) have no yield steps. Rubber materials (curve E) and organic tissues (curve K) can be elongated

substantially in tension. It may be noted that curve K (for organic tissues) begins not from the origin of coordinates, but at a certain initial load F_0 which corresponds to the initial tension existing in organic tissues.

6.2. Specimens for Mechanical Tests

Mechanical properties of materials are determined by testing specimens of these materials which may have round, rectangular or special cross-sectional shape. The shapes and dimensions of test specimens are specified by a state standard. The standard defines their shapes, length and cross-sectional dimensions, dimensions of shoulders for clamping in testing machines, and the direction in which a specimen is to be cut (longitudinal, transverse, tangential or radial).

Round and rectangular specimens for tensile tests (Fig. 141*a* and *b*) have thickened portions, or shoulders, at the ends which are needed for their fastening in the grips of a testing machine. Specimens are manufactured in machine tools. The initial length of a specimen is determined by the formulae:

$$l_0 = 2.82\sqrt{S_0}, \quad l_0 = 5.65\sqrt{S_0}, \quad l_0 = 11.3\sqrt{S_0}$$

Round specimens are mostly made of a diameter $d_0 = 10$ mm. The working surface of specimens is machined to a high surface finish. Sharp edges of rectangular specimens are filed. The surface of specimens of sheet materials is left untouched.

The working surface of manufactured specimens is marked by making a number of small cuts; sheet specimens of a small thickness are marked by a soft pencil.

After marking, specimens are measured to determine the initial length l_0 which should include a number of marks multiple of ten or, for short specimens, a multiple of five. The diameter d_0, width b_0, and thickness a_0 are measured with an accuracy not worse than 0.01 mm. Measurements are done in the mid and at the ends of the working surface in at least three points of the working zone. The smallest of the measured dimensions is used for calculating the cross-sectional area:

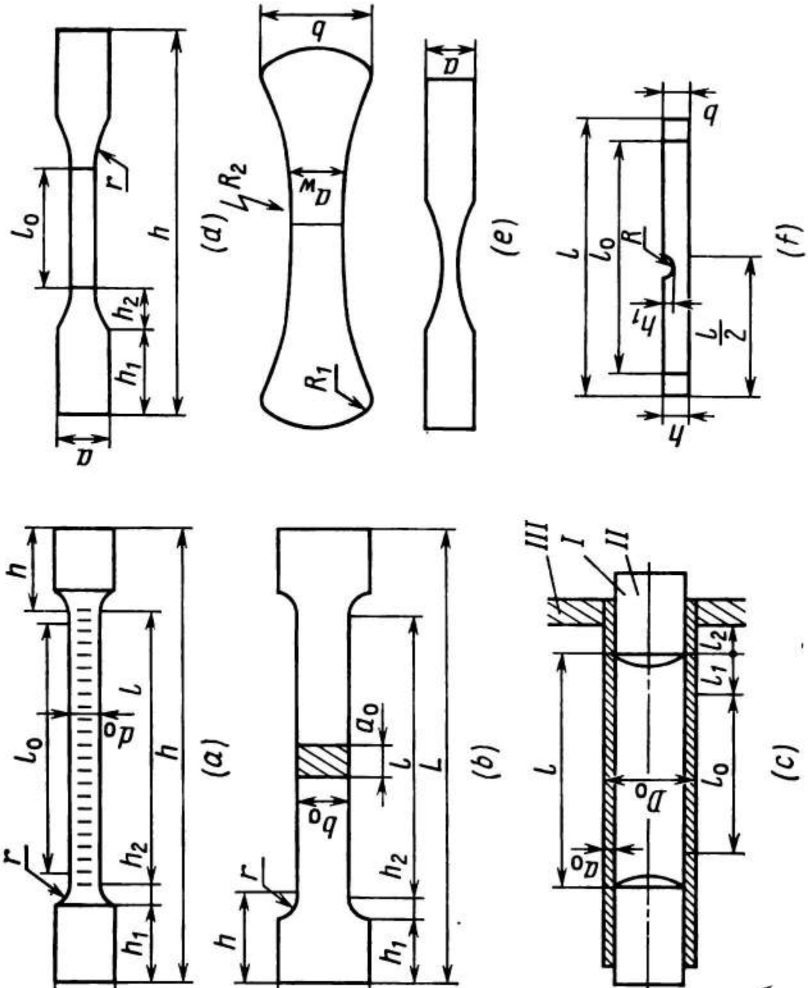


Fig. 141.
Shapes of test specimens

for cylindrical specimens

$$S_0 = \frac{\pi d_0^2}{4}$$

and for rectangular ones

$$S_0 = b_0 a_0$$

Specimens of brittle materials should be marked carefully so as not to damage their surface.

Tests of welded and seamless tubes are specified by a special state standards. The tube specimens should have the initial length (Fig. 141*c*):

$$l_0 = 5.65 \sqrt{S_0} \quad \text{or} \quad l_0 = 11.3 \sqrt{S_0}$$

and the working length

$$l = l_0 + 2l_1$$

With tubes of smaller diameters (D_0 up to 18 mm), test specimens are made by cutting sections from a tube. In that case the cross-sectional area of a specimen is found by the formula:

$$S_0 = \pi a_0 (D_0 - a_0)$$

where a_0 is the thickness of tube wall.

With larger diameter tubes, test specimens are made by cutting strips along the tube axis with the width of the working portion b_0. The cross-sectional area of such a specimen

$$S_0 = K a_0 b_0$$

where K is the coefficient depending on wall thickness and tube diameter.

The working portion of tube specimens is marked as given above.

Specimens for tensile tests of polymer materials are made with a shortened working portion l_0 and enlarged shoulder portion (h_1) for better fastening in machine clamps (Fig. 141*d*). The junctions between the working portion and shoulders are smoother, with the radius r and length h_2.

Specimens for tensile, compressive and bending tests of materials on the basis of organic binders have a thickness not more than 10 mm and are machined at one side only (Fig. 141*e*). With the specimen thickness less than 10 mm, the width of the working portion must be 15 mm and the working width a_w should be equal to the thickness of sheet. The thickness of the working portion is measured in at least three points.

Dynamic (impact) tests of specimens are carried out for determining the impact toughness:

$$a_n = \frac{A_n}{S} = \frac{A_n}{b\,(h-h_1)}$$

where A_n is the impact work, and S is the cross-sectional area of a notched specimen.

A specimen for impact tests is illustrated in Fig. 141*f*. Its principal dimensions are: length l, working length l_0, width b, thickness without a notch h, depth of notch h_1, and notch radius R. A notch is cut in a specimen to facilitate fracture at the desired section.

Machining of test specimens should be done so as to avoid overheating. The surface of specimens is polished. The impact load is applied in tests at the specimen side opposite to the notch.

6.3. Principles of Strain Measurement

Stresses and strains during tests of materials are determined by the methods of strain measurement. The stressed-strained state of a specimen is determined by measuring the relative deformations in the given portion of the specimen and in chosen directions and by calculating the elastic or plastic strains.

Strain measurement in specimens and structures is carried out by means of special devices called strain gauges.

The length of a portion on which a strain gauge is mounted is called the nominal length of strain gauge. It is the distance between the prisms of a mechanical strain gauge or the length of a strain-sensitive portion of a bonded strain gauge.

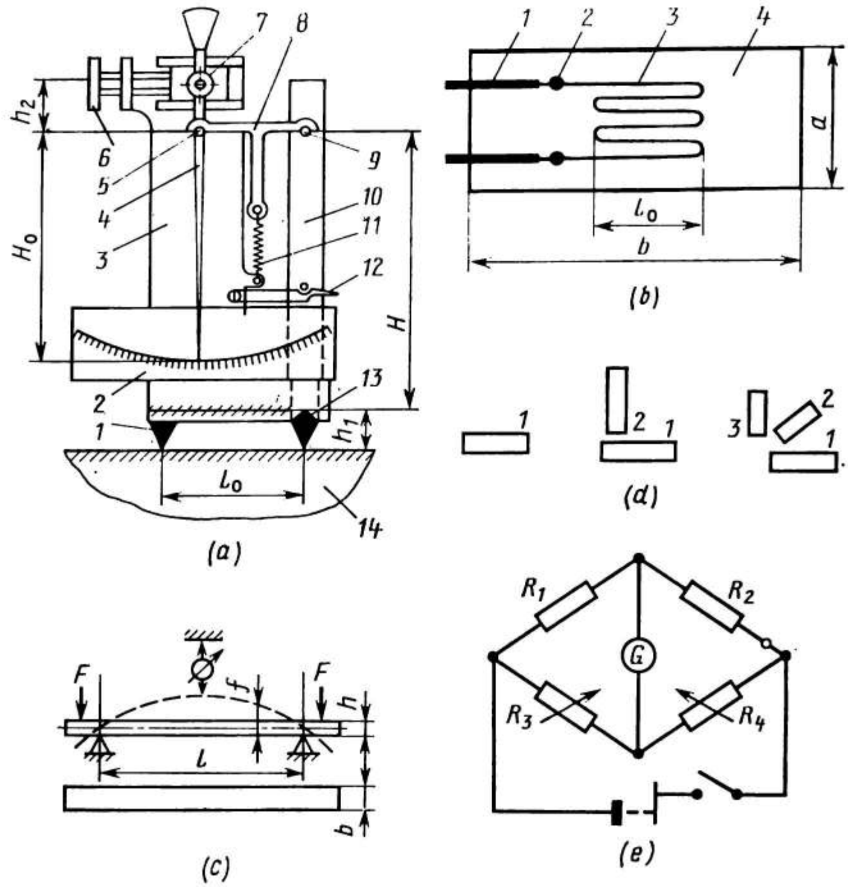


Fig. 142.
Strain measurement in specimens (*a*) scheme of a mechanical lever-type extensometer; *1* — fixed prism; *2* — mirror scale; *3* — housing; *4* — pointer; *5*, *9* — pins; *6* — bearing screw; *7* — bearing; *8* — cross-piece; *10* — lever; *11* — spring; *12* — lever; *13* — movable prism; *14* — specimen; (*b*) scheme of a wire strain gauge; *1* — mounting terminals; *2* — soldered connections; *3* — strain-sensitive element; *4* — base; (*c*) calibration of a strain gauge; (*d*) schemes of glueing of strain gauges; (*e*) scheme of connection of strain gauges

The average strain for the nominal length l_0 of a strain gauge (Fig. 142*a*) is found by the formula:

$$\varepsilon = \frac{\Delta l_0}{l_0} = \frac{\Delta}{m l_0}$$

where Δl_0 is the elongation or contraction of the nominal length of the gauge; Δ is the reading on the gauge scale upon loading; and m is the strain gauge scale.

The scale of a strain gauge is determined as the ratio of the deviation of pointer on the instrument dial to the actual value of strain that causes this deviation.

Experimental determination of the stressed state of specimens and structures can be done by various methods of strain measurement: X-ray, polarization-optical (photoelasticity), method of moiré fringes, methods of brittle coatings, and methods based on electric transformation of strain by means of resistance strain gauges (wire, foil or semiconductor).

The X-ray method of strain measurement is only applicable with materials having a crystalline structure, since it is based on the phenomenon of interference of X-rays passing through the crystal lattice of a material. This method can measure strains only in the elastic region.

The photoelastic (polarization-optical) method uses transparent specimens or models, say, made of acrylic plastic, which are exposed to polarized light. This produces the elasto-plastic effect due to double refraction (birefringence) in the model, so that directions of active stresses become visible even in specimens of an intricate shape.

In the method of moiré fringes, a grid is formed on a specimen, for instance, by photo-chemical etching. Deformation of one of two raster grids in tests produces the effect of moiré fringes.

The method of brittle coatings consists essentially in that a brittle coating, say, lacquer or copper, is applied onto the surface of specimens. In tests, a lacquer coating cracks in zones of the highest stress or strain. In a copper coating, there appear dark spots whose size and intensity vary with an increase of load. Since the thickness of such coatings is very small, it may be taken that the stressed state in the surface of a specimen is the same as that in a coating.

The methods described can determine qualitatively the stressed-strained state and reveal the zones of stress

concentrations. In recent time, holographic methods have also come into use for qualitative determination of stressed states and stress concentration zones.

Practical tests of specimens and structures are mostly carried out by using mechanical, optical, and electrical (resistance) strain gauges of the wire, foil or semiconductor type.

A Hugenberg mechanical lever-type strain gauge (see Fig. 142*a*) has a variable nominal length. Its operating principle is based on scaled magnification of the deformation by means of a mechanical transmission to a value that can be measured reliably. The ultimate magnitude of the measured elongation of the gauge length is up to 0.2 mm. The standard nominal length of the gauge may be 10 or 20 mm or, with the use of extension pieces, 100, 200, or 1000 mm. The magnification of the gauge is:

$$\frac{H}{h_1}\frac{H_0}{h_2} = 100\text{-}2000$$

The design of the gauge is essentially as follows (see Fig. 142*a*). Housing *3* has in the bottom portion a cross-piece with a fixed prism *1* and a recess for movable prism *13*. The latter is attached at the end of a two-arm lever *10* whose second arm carries a pin *9*. A pointer *4* arranged in front of a mirror dial *2* turns on an axis in the top portion of the gauge housing. Lever *10* and pointer *4* are connected by means of a cross-piece *8* which is pressed against pins *5* and *9* by spring *11*. The position of the pointer relative to the dial can be controlled by means of an adjusting screw *6* which moves the bearing support *7*.

The gauge can be arranged on a specimen *14* in any desired position by means of clamps, cramps, cemented shoes, welded ties, suction cups, or magnets. Deformations are measured only in static loading.

The operating principle of wire, foil and semiconductor resistance strain gauges is based on variations of the electric resistance of some metals and semiconductors under load (see Fig. 142*b*). A resistance strain gauge has a substrate *4* made of paper, lacquer film or cloth serving as insulation material, a looped wire or foil *3*, semi-

conductor plates, and mounting terminals *1* which are soldered to the wire or foil. The gauge is fastened in place by various binders or cements. The dimensions of the substrate, i.e. the width a and length b, are chosen depending on the nominal length l_0.

Resistance strain gauges differ, for instance, from the mechanical type in that their strain-sensitive element is mechanically bonded to the specimen all over its surface. The strain sensitivity of a resistance strain gauge is characterized by the ratio of the variation of its resistance under strain to the magnitude of relative strain.

With the initial resistance R_0 of the wire and the initial nominal length l_0, a deformation causes an increase of the length by Δl and a change of resistance by ΔR, so that

$$\frac{\Delta R}{R_0} = K \frac{\Delta l}{l}$$

or

$$\frac{\Delta R}{R_0} = K\varepsilon$$

As may be seen from these formulae, the relative change of resistance of a strain gauge is directly proportional to its deformation. The coefficient K is called the gauge factor of a strain gauge. It is determined experimentally on a calibration beam.

The gauge factor is determined by the formula:

$$K = \frac{\Delta R}{R_0 \varepsilon}$$

The nominal length of resistance strain gauges may range from a few tenths of a millimetre to a few tens of millimetres. Wire strain gauges with the nominal length of 5 mm, 10 mm and 20 mm are most popular.

Wires of wire strain gauges are usually made of constantan which can operate at temperatures up to 350°C, or Ni-Cr alloys (up to 900°C). The wire diameter is 10-30 μm. The suitable materials for semiconductor strain gauges are silicon and germanium single crystals and bismuth polycrystals.

Foil strain gauges have a grid of a thin-sheet material (foil) of a thickness of 5-10 μm. The substrate is a synthetic resin film or glue-impregnated paper. The thickness of a resin-film substrate is 30-40 μm and that of a paper substrate, 80-100 μm. Electric terminals of strain gauges are made from copper wire of a diameter of 0.12-0.15 mm. A pattern is formed on the foil by contact printing from a negative covered by a light-sensitive acid-resistant composition. After printing, the foil portions having no coating are etched off and mounting terminals are fastened.

In semiconductor resistance strain gauges, the sensitive element is a monocrystalline semiconductor of a thickness of 20-50 μm, width up to 0.5 mm, and length 2-12 mm. A semiconductor single crystal is cut into plates which are then etched to eliminate microfissures produced by machining.

Manufactured strain gauges are calibrated on a calibration steel beam of constant cross section (Fig. 142*c*) or cantilever beam in order to determine their gauge factor K.

A beam of constant cross section (see Fig. 142*c*) is loaded at the ends by equal forces. The beam has the following dimensions: width b, thickness h, and length l between supports.

The beam is bent under the action of forces F and its working portion of the length l is deformed. This deformation is determined by the deflection f in the mid of beam:

$$\varepsilon_{\mathrm{cal}} = 4f\,\frac{h}{l^2}$$

The deflection is measured by a dial indicator. When strain gauges are bonded on a specimen, they are oriented in directions where the highest normal stresses are most probable. A single strain gauge is bonded for a uniaxial stressed state, two for biaxial stressed state, and several gauges, for combined stressed states.

In order to determine the magnitude and direction of main deformations, it is sufficient to make strain measurement in three directions. If the directions of

main deformations are known, it suffices to measure the deformations in two mutually perpendicular directions in order to determine the maximum and minimum deformations.

In the general case, the magnitude of main deformation can be found by the equation:

$$\varepsilon_\varphi = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2} \cos 2\varphi + \frac{\gamma_{xy}}{2} \sin 2\varphi$$

where ε_x, ε_y, and γ_{xy} are respectively the longitudinal, lateral and shear deformation, and φ is the angle between resistance strain gauges.

For two resistance strain gauges bonded at an angle of 90° relative to each other, deformations can be found by the formulae:

the maximum deformation

$$\varepsilon_{\max} = \varepsilon_1 + \mu\varepsilon_2$$

the minimum deformation

$$\varepsilon_{\min} = \varepsilon_1 - \mu\varepsilon_2$$

and the shear deformation

$$\gamma_{\max} = \varepsilon_1 - \varepsilon_2$$

where μ is Poisson's ratio, and 1 and 2 are the numbers of bonded strain gauges.

The measured deformations can be recalculated into equivalent mechanical stresses by the formulae:

for the maximum normal stresses

$$\sigma_{\max} = \frac{E}{1-\mu^2} (\varepsilon_{\max} + \mu\varepsilon_{\min})$$

for the minimum normal stresses

$$\sigma_{\min} = \frac{E}{1-\mu^2} (\varepsilon_{\min} + \mu\varepsilon_{\max})$$

and for shear stresses:

$$\sigma_{\max\,\gamma} = \frac{E}{2(1+\mu)} \gamma_{\max}$$

where E is the modulus of elasticity, and μ is Poisson's ratio.

If the measured deformations ε_1 and ε_2 in a homogeneous isotropic material are within the elastic limit, the main stresses σ_1 and σ_2 can be found by the formulae:

$$\sigma_1 = \frac{E}{1-\mu^2}(\varepsilon_1 + \mu\varepsilon_2)$$

$$\sigma_2 = \frac{E}{1-\mu^2}(\varepsilon_2 + \mu\varepsilon_1)$$

The results of strain measurement can be processed by electronic computers and automatic complexes.

On specimens for tensile and compressive tests, strain gauges should be arranged as close as possible to the specimen axis. In bending tests, on the contrary, they should be placed as far as possible from the central axis so as to measure the largest deformations.

In combined loading, for instance, in a combined action of a longitudinal force and bending moment, two strain gauges are arranged on opposite sides of a specimen at equal distances from the central axis and in the latter's plane. The longitudinal stresses will then be determined by the sum of readings of the gauges and the bending moments, by their difference.

The torsional moment in torsional tests of a round specimen is determined by means of one strain gauge arranged at an angle of 45° to the generatrix or by two gauges arranged at an angle of 45° to the generatrix and at 90° to each other.

Static deformations are measured by means of resistance strain gauges connected into a balanced bridge circuit (Fig. 142*e*). The circuit comprises a working resistance strain gauge R_1, 'dummy' strain gauge R_2, and two controlled resistors R_3 and R_4.

Measurements are done by means of a galvanometer. The 'dummy' strain gauge R_2 is bonded onto a similar specimen of the same material as the tested one, but not subjected to loading. The controlled resistors R_3 and R_4 are built-in in the instrument. The signal produced by a resistance strain gauge can be amplified by an electric amplifier and recorded by means of a recorder.

On application of a load, the specimen is deformed and the resistance R_1 changes, which causes disbalance

in the bridge circuit. The bridge is brought back to balance by adjusting the resistor R_3 until the pointer of the galvanometer G returns to the zero position. In a balanced bridge circuit, a change of resistance R_3 is proportional to the relative elongation, which makes it possible to calibrate the galvanometer directly in units of relative strain.

6.4. Design of Testing Machines

Stress-strain diagrams of materials are obtained in laboratories and at industrial works in all-purpose and special testing machines with static or dynamic loading.

A specimen is subjected in a testing machine to a controlled tensile load, its elongation is measured, and the stress-strain diagram is recorded automatically.

Static tests are carried out in testing machines of two types: with a constant rate of deformation or with a constant rate of loading. Machines of the first type are used for testing metals, wood and fabrics and those of the second type, for testing, for instance, cement specimens.

Testing machines have a hydraulic or mechanical drive.

Modern all-purpose testing machines with hydraulic drive can develop a force from 100 kN to 2000 kN and special ones, up to 5000 kN.

The force applied is measured by means of levers with movable weights, pendulums, hydraulic devices, and torsion bars. The load is measured by pressure gauges or spring-type dynamometers.

A testing machine for static tensile tests with a force up to 1000 kN is shown schematically in Fig. 143.

The machine is fastened to the foundation by anchor bolts *1* and comprises a frame *3*, hydraulic station *18*, guide columns *10*, lower cross-piece *4*, intermediate cross-piece *9*, upper cross-piece *13* with grips *5* and *8* for specimen *6*, and resistance strain gauge unit *7*.

The force-measuring unit of the machine has circular reading scales *15* and diagram-recording unit *16*. The machine is controlled from a control board *17*.

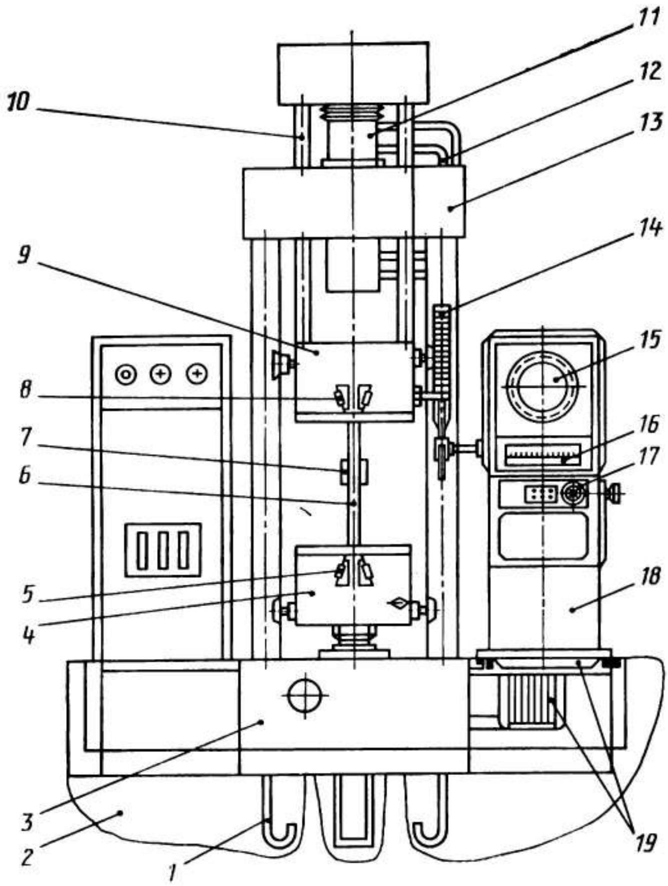


Fig. 143.
Scheme of a universal tensile-testing machine *1* — anchor bolts; *2* — foundation; *3* — base frame; *4* — lower cross-piece; *5* — lower grip; *6* — specimen; *7* — strain gauge block; *8* — upper grip; *9* — intermediate cross-piece; *10* — guide columns; *11* — hydraulic cylinder; *12* — pipelines; *13* — upper cross-piece; *14* — elongation scale; *15* — force-measuring scale; *16* — recorder; *17* — control board; *18* — hydraulic station; *19* — pipelines

The machine operates as follows. A specimen *6*, with or without the strain gauge unit, is fastened in grips *5* and *8*. On turning the handwheel on the control board *17*, oil under the working pressure is admitted into hydraulic cylinder *11* with a piston which moves the intermediate cross-piece *9* together with the upper grip *8*. The load is

determined by the pressure of oil in the cylinder, which is measured by a dynamometer. An increase of oil pressure in the cylinder is indicated on the force-measuring scale *15*. During the test, the stress-strain diagram is recorded automatically in recording unit *16* on a strain scale 1:1, 20:1 or 100:1 and with the maximum ordinate up to 320 mm.

The highest load in machines of this type is 1000 kN. There are three reading scales with the lowest division value 400 N, medium value 1000 N, and the highest division value 2000 N. On rupture of a specimen and pressure drop in the hydraulic system, the rupture load is fixed by a control pointer.

The kinematic characteristics of the machine are as follows: the largest spacing between clamps, including the piston stroke, is equal to 1100 mm; the speed of movable clamp *8* without load is 0-100 mm/min; and the maximum stroke of piston is 340 mm.

All-purpose testing machines are employed for tests of specimens in tension, compression, bending, and a cyclic loading (fatigue tests).

Non-metallic materials, such as specimens of asbestos-cement tubes, are tested for compression in testing presses as that shown in Fig. 144.

The press has a load scale *1*, diagram recorder *2*, hydraulic station *3*, oil pipelines *5*, frame *6*, table with appliances *7*, loading plate *8*, upper clamp *9*, upper plate *10*, handwheel *12*, screw *13*, cross-piece *14*, and adjustment roller *16*. The press is mounted on foundation *4*.

The loading arrangement is a closed frame with a device *7* for the supply and placing of test tubes *15*. The test tube is pressed by upper clamp *9*. The mechanical load is applied to a specimen by means of a hydraulic drive supplied with oil from hydraulic station *3* through pipelines *5*. The force is measured by a torsion bar and indicated on a circular scale *1* and the stress-strain diagram ($F - \Delta l$) is recorded automatically by recorder *2*.

Testing presses are manufactured for the maximum load of 100, 500, 1250, 2500, and 5000 kN, with the width of the testing section ranging from 250 mm to 730 mm, the piston stroke 50 mm, and the rate of motion of the

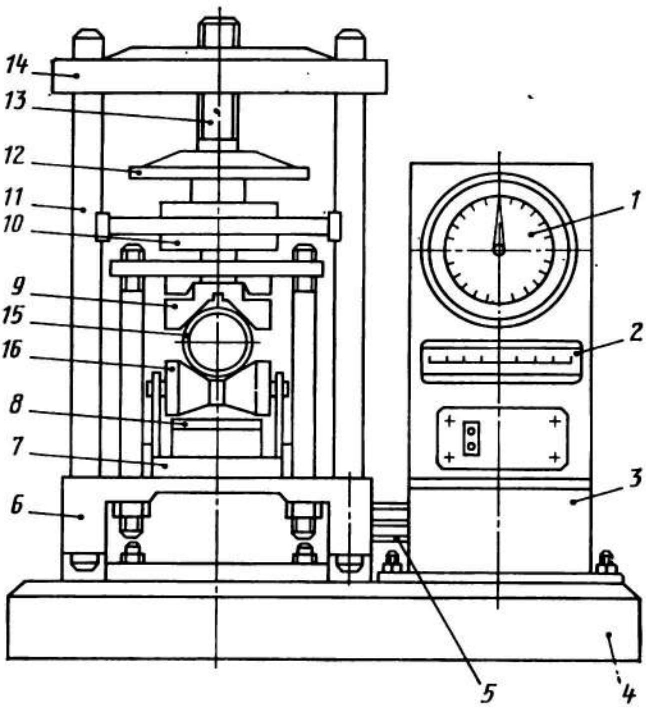


Fig. 144.
Scheme of a hydraulic testing press
1 — load scale; *2* — recorder; *3* — hydraulic station; *4* — foundation; *5* — pipelines; *6* — frame; *7* — table with appliances; *8* — plate; *9* — upper clamp; *10* — plate; *11* — column; *12* — fly-wheel; *13* — screw; *14* — cross-piece; *15* — specimen; *16* — adjusting roller

movable cross-piece 200-300 mm/min.

Standard specimens are tested in universal clamps. In some cases, special clamps are needed. All clamps are provided with additional arrangements for specimen fastening: chain pins, ring brackets, spring hooks, clamping flanges for compression tests of springs.

Chapter Seven

Mechanical Tests of Specimens and Structures

7.1. Hardness Measurements

When selecting a suitable material, the designer first of all considers its hardness which is a rough criterion of ultimate strength.

Hardness is understood as the property of a material to resist indentation of a harder body which acquires no residual deformation.

Methods of hardness measurement are mostly based on indenting a working body (indenter) of a particular shape into the surface of the specimen whose hardness is to be tested. The indenter has a much higher hardness than the specimen and its shape and dimensions are strictly specified. It is also specified whether a static or dynamic load is to be applied.

Hardness is usually measured in static loading. The load is applied to the indenter slowly and gradually during a specified time interval. The main methods of hardness measurement are the Brinell test using a hardened steel ball; Rockwell test using a diamond cone or a hardened steel ball of a very small size; and Vickers test in which a four-faced diamond pyramid is employed.

In the Brinell test (Fig. 145*a*), the hardness of a specimen is measured on indenting a hardened steel ball of a diameter of 2.5 mm, 5 mm or 10 mm. The thickness of the specimen must be at least tenfold of the indentation depth, i.e. $h_0 = 10h$. The thickness of specimens and the holding time are chosen depending on the properties of the material to be tested.

An instrument for hardness measurement by indentation of a ball is shown in Fig. 145*d*. A holder with ball *7* is attached to a spindle *6*. A specimen is placed on a table *8* which can be moved vertically by screw *10*. On turning a handwheel *9*, the specimen is pressed against the ball. The centre of contact with the ball and therefore

the centre of future imprint should be at a distance not less than the ball diameter from the specimen edge and the centre of an adjacent imprint, at a distance not less than two ball diameters.

Before testing, the ball should be wiped dry. The load is applied perpendicular to the specimen plane; shocks

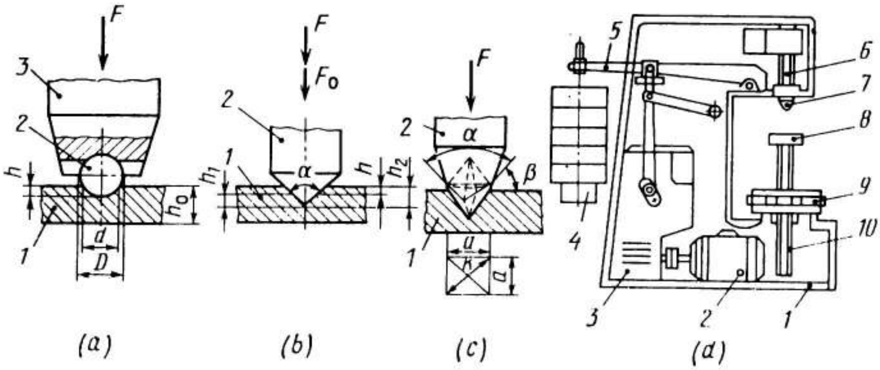


Fig. 145.
Various methods for hardness testing of materials
(*a*) Brinell test (*1* — specimen; *2* — ball; *3* — ball holder); (*b*) Rockwell test (*1* — specimen; *2* — diamond cone); (*c*) Vickers test (*1* — specimen; *2* — diamond prism); (*d*) hardness testing apparatus (*1* — housing; *2* — electric drive; *3* — reduction gear; *4* — weight; *5* — loading lever; *6* — spindle; *7* — holder with ball; *8* — specimen table; *9* — flywheel; *10* — screw)

and impacts must be avoided. The load is increased gradually to the specified value and a holding time is allowed (15 s for hard materials and 30 s for softer ones). The load is then released and the imprint produced by the ball is measured in two mutually perpendicular directions by means of a microscope or special gauge.

Imprints of balls of a diameter of 5 mm and 10 mm should be measured with an accuracy of ±0.05 mm and those of 2.5-mm balls, with an accuracy of ±0.01 mm.

The hardness measured by ball indentation (Brinell hardness) is determined as

$$HB = \frac{F}{S}$$

where F is the force applied, and S is the area of the spherical imprint which can be calculated by the formula:

$$S = \frac{\pi D^2}{2} - \frac{\pi D}{2} \sqrt{D^2 - d^2}$$

where D is the diameter of the ball, mm, and d is the diameter of the imprint, mm.

Substituting the area into the previous formula, we get

$$HB = \frac{F}{\frac{\pi D^2}{2} - \frac{\pi D}{2} \sqrt{D^2 - d^2}}$$

or, after transformations:

$$HB = \frac{F}{D^2} \left(\frac{2/\pi}{1 - \sqrt{1 - (d/D)^2}} \right)$$

Denoting $F/D^2 = K$, we finally have:

$$HB = K \left(\frac{2/\pi}{1 - \sqrt{1 - (d/D)^2}} \right)$$

The hardness of a material, as measured by balls of different diameters, will be constant if K is constant.

For correct choice of load and ball diameter, it should be observed that

$$0.2D < d < 0.6D$$

The Brinell hardness HB may serve for rough estimation of the ultimate tensile strength of a material:

$$\sigma_t = 1/3HB \approx 0.35HB$$

The result of Brinell hardness test is designated by HB with appropriate digits. For instance, with the ball diameter $D = 10$ mm and the load $F = 29.4$ kN applied for 10 s, the Brinell hardness number will be 260 HB. With different test conditions, another form of record is possible. For instance, in a test with a ball of a diameter $D = 5$ mm and load 7.35 kN applied for 20 s, the

designation will be 341 *HB* 5/7, 35/20. The minimum thickness of specimens for Brinell hardness tests is chosen according to the data of Table 6.

Table 6

Material	*HB*	h_0, mm	$K = \frac{F}{D^2}$	*D*, mm	*F*, kN	τ, s
Steel and cast iron	140-450	6-3 4-2 <2	30	10 5 2.5	29.40 7.35 1.84	10
Non-ferrous metals	130	6-3 4-2 <2	30	10 5 2.5	29.40 7.35 1.84	30
Aluminium alloys	8-35	>6 6-3 <3	2.5	10 5 2.5	2.45 0.61 0.15	60

The essence of the Rockwell hardness test (Fig. 145*b*) consists in that an indenter (steel ball or diamond cone) is pressed into the surface of a specimen. Rockwell hardness is measured in conditional units which simplifies calculations. In this method, the procedures of indentation of a cone or ball and of measurement of the imprint are combined in time.

Steel balls used in the Rockwell test have a diameter $D = 1.588$ mm. The diamond cone has an apex angle $\alpha = 120°$ and the end is rounded off to a radius $r = 0.2$ mm. The diamond cone is used for testing specimens of hard hardened steels.

The specimen to be tested should have a flat polished surface. Curvilinear specimens should have the radius of curvature not less than 15 mm; otherwise a flat should be ground in the surface.

The minimum thickness of specimens should be at least eight times the depth of indentation (Table 7). The support (bottom) surface of a specimen should lie tightly and firmly on the testing table of the apparatus.

Table 7

Hardness number	Minimal thickness of specimen, mm, for hardness test on scale		
	A	B	C
20	—	—	1.5
25	—	2.0	1.4
30	—	1.9	1.3
40	—	1.7	1.2
50	—	1.5	1.0
67	—	1.2	0.7
70	0.7	1.2	—
80	0.5	1.0	—
90	0.4	0.8	—
100	—	0.7	—

The indenter is pressed into the surface of a specimen under the action of two loads applied successively: the preliminary load $F_0 = 98$ N which presses the indenter to a depth h_1, and the main load $F = 980$ N for a steel ball or $F = 1470$ N for a diamond cone, which causes the indenter to penetrate to a greater depth h. Thus, the depth of indentation under the action of both loads, F_0 and F, is:

$$h_2 = h + h_1$$

The depth of indentation is read off the apparatus scale after releasing the main load and leaving the specimen under the initial (preliminary) load. The apparatus has three scales: A, B, and C. In tests of heat-treated steel specimens by the diamond cone, the indentation depth is measured on the scale C and the result is designated by the letters *HRC*. The highest load for the C scale is 1470 N.

The scale A with the highest load 588 N is used for testing very hard materials, thin-sheet specimens and for determining the hardness in surface-hardened specimens. The respective designation is *HRA*.

The B scale is employed in hardness tests with the steel ball with the highest load 980 N; the designation is *HRB*.

The indicator scale has 100 divisions each equal to 0.002 mm. This value is adopted as the unit of hardness and corresponds to the axial movement of the indenter by 0.002 mm.

Rockwell hardness numbers (*HR*) are expressed by the following formulae:

for scales A and C

$$HRC = 100 - \frac{h_2 - h_1}{0.002}$$

and for scale B

$$HRB = 130 - \frac{h_2 - h_1}{0.002}$$

where h_2 is the depth of indentation into a specimen under the total load; h_1 is the depth of indentation under the action of the preliminary load; 100 or 130 is the number of scale divisions, and 0.002 is the value of a division.

Rockwell hardness numbers can be converted into Brinell hardness numbers and vice versa by using special tables.

The Vickers method of hardness tests is based on pressing an indenter in the form of a four-faced diamond pyramid into the surface of a specimen.

In the diamond pyramid used for the purpose (Fig. 145*c*), the angle between opposite faces is $\alpha = 136°$ and the angle made by a face with the specimen surface, $\beta = 22°$. When this pyramid is used, the angle of indentation is always the same for any size of the imprint. The indicated angle α has been adopted in order to match the Vickers hardness numbers with the Brinell and Rockwell numbers.

The thickness of specimens for Vickers hardness tests should be not less than $1.5d$ for non-ferrous metals and not less than $1.2d$ for steels (where d is the imprint diagonal). The diagonals of an imprint are measured under a microscope with an accuracy of not worse than ± 0.001 mm.

Vickers hardness tests are carried out on stationary and portable apparatus.

In tests, the load is chosen multiple of 5 kgf (from 5 to 100 kgf, or from 49 to 980 N). The pyramid is preliminarily arranged at 0.1-0.3 mm above the surface of a specimen. The load is applied for 10-15 s; load application is signalled by illumination of a lamp.

After load releasing, the axis of the microscope is placed at the centre of rhombic imprint to measure the imprint diagonals, and then the hardness number for the particular load applied is found in the tables.

Imprints should be made on a specimen so that the distance from the imprint to the edge of specimen or the edge of another imprint be not less than 2.5 imprint diagonals.

Imprint diagonals should be measured under the microscope with an accuracy of 0.001 mm if the diagonal is less than 0.2 mm or with an accuracy of 0.002 mm if it is greater than 0.2 mm. In tests, the hardness at a load of 49 N is first determined approximately, after which the largest allowable diagonal of imprint for the specimen of given thickness is found from the relationship $d = h_0/3$. For this diagonal and the approximate hardness number, the highest possible load is found from conversion tables.

The Vickers hardness HV is calculated by the formula:

$$HV = \frac{F}{S}$$

where F is the load, N, and S is the area of imprint of the pyramid, mm^2

$$S = \frac{d^2}{2 \sin \frac{\alpha}{2}}$$

where d is the mean arithmetic length of the two diagonals, mm:

$$d = \frac{d_1 + d_2}{2}$$

and α is the apex angle of the pyramid between two opposite faces. Substituting for S into the formula of Vickers hardness, we obtain:

$$HV = \frac{2F}{d^2} \sin \frac{\alpha}{2}$$

or, upon substituting the numerical value of sin α:

$$HV = 0.189 \frac{F}{d^2}$$

Recommended loads depending on specimen thickness and HV number are given in Table 8.

Table 8

Specimen thickness, mm	Loads, N, at HV hardness			
	25-50	50-100	100-300	300-900
0.3-0.5	—	—	—	5-10
0.5-1	—	—	5-10	10-20
1-2	5-10	5-10	10-20	10-20
2-4	10-20	20-50	20-50	20-50
4-6	20-50	30-50	50	50-120

Apart from the hardness tests described above, other methods are also in use, such as the method of elastic striker, ball impact method, etc.

7.2. Determination of Modulus of Elasticity and Poisson's Ratio

The stress-strain diagrams of tested materials have an initial linear portion of the curve (see Fig. 139). The angle of incline of this portion to the coordinate axes is different for various materials. It should then be clear that the angle of incline of the curve is the measure of strain of a material at a given stress, i.e. the measure of elasticity or yieldability of that material.

By Hooke's law

$$\sigma = E\varepsilon$$

whence

$$E = \sigma/\varepsilon$$

This ratio is called the modulus of elasticity, or Young's modulus. The strain ε in tension is determined by the formula:

$$\varepsilon = \frac{\Delta l}{l}$$

and the active stress, by the formula:

$$\sigma = \frac{F}{S_0}$$

Substituting these expressions into the formula for elastic modulus, we obtain:

$$E = \frac{F/S_0}{\Delta l/l} = \frac{Fl}{S_0 \Delta l}$$

where $l/\Delta l = 1/\varepsilon$ is an inverse of strain.

Thus, in tensile tests of specimens, it is required to find the elongation Δl, whereas the initial cross-sectional area S_0 and the initial length l of the working portion of the specimen are known.

In tensile tests of specimens in a rupture test machine, elongations Δl are measured by means of mirror extensometers (Fig. 146*a*). Two mirrors, *4* and *8*, fastened in a holder *2*, are mounted by means of a clamp *3* on a specimen to be tested, *1*. The mirrors are illuminated by light beams directed from extensometer tubes *6* and *9*. As the length of the specimen increases under load, this is indicated by the mirrors on reading scales *5* and *7*. The elongation is calculated as the arithmetic mean of the readings of two mirrors.

The elongation of the design portion of a specimen is found by the formula:

$$\Delta l = \frac{ac}{2h}$$

where a is the greater diagonal of the extensometer prism; c is the arithmetic mean of the variations read on the scale:

$$c = \frac{c_1 + c_2}{2}$$

and h is the distance between a mirror and scale.

The distance h is chosen so as to satisfy the equality:

$$\frac{a}{2h} = \frac{1}{500}$$

The elastic modulus of various materials is different. For instance, it is equal to 7 MPa for rubber, 1400 MPa for nylon, 14 000 MPa for wood along fibres, 70 000 MPa

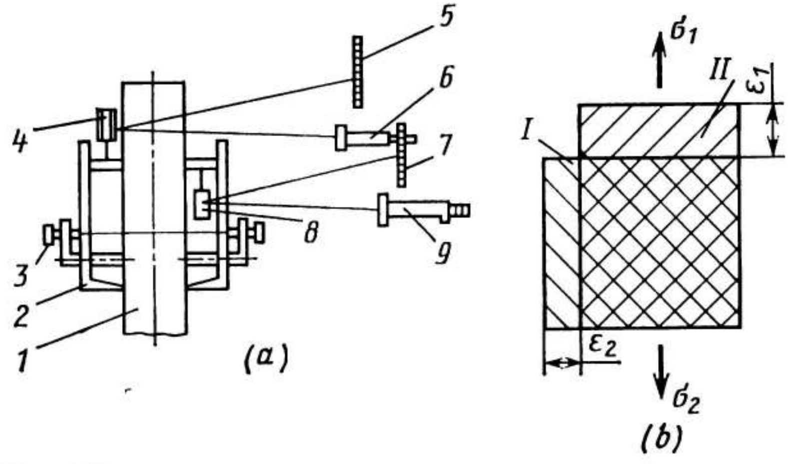


Fig. 146.
Scheme of an apparatus for measuring the modulus of elasticity (*a*) and loading diagram of a plate (*b*)

1 — specimen; *2* — holder; *3* — grip; *4* and *8* — mirrors; *5* and *7* — measuring scales; *6* and *9* — extensometers

for aluminium alloys, 210 000 MPa for steels, and as high as 1 200 000 MPa for diamond.

The modulus of elasticity of isotropic materials is determined in a simple way by tensile tests.

More intricate methods have been developed for determining the elastic modulus of materials having an anisotropic structure, such as polymers and wood.

Specimens for tests of sheet and laminated plastics are made in the form of bars of rectangular cross section. They are tested in tension and bending. Specimens have a length of 300 mm, width $b = 30$ mm, and thickness $a = 20$-30 mm. Specimens are cut from a material in three directions: longitudinal, transverse, and at an angle of 45°. The number of specimens should be not less than three for each direction. In tests, a specimen is loaded several times.

The load increment

$$\Delta F = F_{\max} - F_0$$

The elastic modulus in a tensile test with the i-th loading is calculated by the formula:

$$E_i = \frac{\Delta F}{S_0} \frac{l}{\Delta l}$$

where ΔF is the load increment; l is the nominal length of strain gauge; Δl is the mean arithmetic elongation calculated by the results of three measurements:

$$\Delta l = \frac{\Delta l_1 + \Delta l_2 + \Delta l_3}{3}$$

and S_0 is the initial cross-sectional area of the specimen:

$$S_0 = b_0 h_0$$

In bending tests, the elastic modulus is calculated by the formula:

$$E_i = \frac{\Delta F}{\Delta f} \frac{L^3}{4 b_0 h_0^3}$$

where L is the span between supports; Δf is the arithmetic mean of deflection of the specimen calculated by three measurements (see formula for Δl).

The design modulus of elasticity in both cases is determined by the formula:

$$E_d = \frac{\sum_i E_i}{n}$$

where n is the number of tested specimens.

Thus, the elastic modulus is expressed by the relationship:

$$E = \frac{\sigma}{\varepsilon}$$

Let us now consider experimental determination of Poisson's ratio. Let a tensile load be applied to one side of a specimen in the form of a square plate (Fig. 146*b*). This kind of loading is called uniaxial. Before application of the load, the plate is square (position *I*), but upon application of the tensile load, which produces a stress σ_1 in the plate, the latter is elongated in the elastic region and becomes rectangular, with the larger side along the action of active stress σ_1 (position *II*).

The expression for ε_1 in the direction of tension is

$$\varepsilon_1 = \frac{\sigma_1}{E}$$

On the other hand, the plate is contracted transversely (strain ε_2). Its corners, however, remain rectangular.

If a number of specimens of various materials are taken and subjected to load, it will be found that the ratio of strains ε_1 and ε_2 is the same in all cases. This is what is called Poisson's ratio μ:

$$\frac{\varepsilon_2}{\varepsilon_1} = \mu$$

The strain ε_1 in the direction of tensile stress is called the longitudinal, or primary, strain. It is caused by tensile stresses and is therefore positive. The strain ε_2 is the lateral, or secondary strain; it is negative.

Strictly speaking, Poisson's ratio has a negative sign which is however disregarded in calculations. It may be written for the secondary strain that

$$\varepsilon_2 = \mu \varepsilon_1$$

Substituting for ε_1 from Hooke's law, we obtain:

$$\varepsilon_2 = \mu \frac{\sigma_1}{E}$$

Thus, if μ and E are known, it is possible to find ε_1 and ε_2. In a biaxial stressed-strained state with stresses

σ_1 and σ_2 (planar problem), the expression for the total strain in the direction of stress σ_1 is

$$\varepsilon_1 = \frac{\sigma_1 - \mu\sigma_2}{E}$$

and that for the total strain in the direction of stress σ_2

$$\varepsilon_2 = \frac{\sigma_2 - \mu\sigma_1}{E}$$

Poisson's ratio of hard materials, such as metals, stone, and concrete, ranges between 1/4 and 1/3.

The determination of the elastic modulus G in shear and torsion will be discussed in Sec. 7.5.

7.3. Impact Tests

Structures can experience in operation both static and dynamic loads. In the former case, a load is applied gradually during an appreciably large length of time and the strain energy in the material also increases gradually. In the latter case, a load is applied actually instantaneously and the stresses and strains appearing in the materials are substantially higher than in the former case.

In impact tests, the dynamic load can be applied longitudinally, in bending or in torsion.

If a weight G falls from a height h onto a specimen, the absolute dynamic elongation can be determined in terms of the static elongation by the formula:

$$\Delta l_d = \Delta l_{st} + \sqrt{\Delta l_{st}^2 + 2\Delta l_{st} h}$$

Dividing the expression under the root sign by Δl_{st}, we get:

$$\Delta l_d = \Delta l_{st} + \Delta l_{st} \sqrt{1 + \frac{2h}{\Delta l_{st}}}$$

This formula can be re-written as follows:

$$\Delta l_d = K \, \Delta l_{st}$$

where K is the dynamic coefficient:

$$K = 1 + \sqrt{1 + \frac{2h}{\Delta l_{st}}}$$

Let us recall that Hooke's law gives the following expression for elongation:

$$\Delta l_{st} = \frac{Gl}{ES_0}$$

The dynamic stress is defined as

$$\sigma_d = \varepsilon_d E$$

where

$$\varepsilon_d = \frac{\Delta l_d}{l}$$

Substituting for Δl_d, we obtain the formula of dynamic stress

$$\sigma_d = E\,\frac{\Delta l_d}{l} = E\,\frac{\Delta l_{st} + \sqrt{\Delta l_{st}^2 + 2\Delta l_{st} h}}{l}$$

If h is sufficiently large compared with Δl_{st}, the squared term under the root sign can be neglected:

$$\sigma_d = E\,\frac{\Delta l_{st} + \sqrt{2\Delta l_{st} h}}{l}$$

If the weight is applied instantaneously, then $h = 0$ and the second term under the root sign turns to zero. Extracting the root, we have:

$$\Delta l_d = \Delta l_{st} + \Delta l_{st} = 2\,\Delta l_{st}$$

The dynamic stress will then be:

$$\sigma_d = E\,\frac{\Delta l_d}{l} = 2E\,\frac{\Delta l_{st}}{l} = 2\sigma_{st}$$

In the case of bending under an impact load, the dynamic stress is determined by taking the ratio of the static and dynamic deflection:

$$\sigma_d = \frac{El}{4W}\left(1 + \sqrt{1 + \frac{2h}{f_{st}}}\right) = \frac{Fl}{4W}\,K$$

If $h = 0$, then $K = 2$, i.e. the dynamic stress will be twice the static stress.

Let us consider impact tests of notched specimens. A notched specimen is a short bar of rectangular cross

section with a notch made in its mid. The notch has the length of the linear portion 2 mm and the rounding radius 1 mm (Fig. 147*a*). Several specimens are cut, say, from a rolled section in the direction of rolling of the metal and perpendicularly to it. All angles of a specimen should be right and the axis of notch, perpendicular to the longitudinal axis of the specimen.

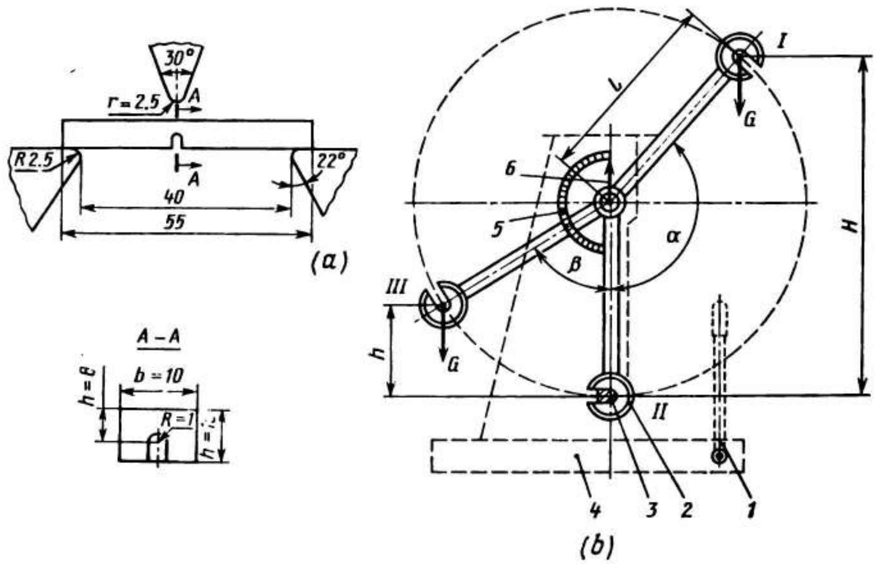


Fig. 147.
Diagram of specimen loading (*a*) and scheme of an impact-testing machine (*b*)

1 **— lever; *2* — pendulum; *3* — specimen; *4* — housing; *5* — measuring scale; *6* — pointer**

Notches are cut by milling, drilling or grinding with an emery disc. The surface of specimens is polished.

A specimen is placed in an impact testing machine, say, a Charpy impact machine. The scheme of impact loading and respective designations are given in Fig. 147*b*. The test procedure is as follows. Specimen *3* is mounted on supports (see Fig. 147*a*), and pendulum *2* of a weight G is moved upwards through an angle α into position *I* where it is held by a catch. A pointer *6* which is mounted on the same shaft with the pendulum, is set before the test vertically, i.e. to zero on scale *5*. As the catch is

disengaged, the pendulum falls and strikes specimen *3* at the side opposite the notch (position *II*). Upon striking the specimen, the pendulum moves further and deviates from the vertical axis through a certain angle β into position *III*. The angle β is read off scale *5* when the pendulum comes back into the vertical position *II*.

The rocking plane of the pendulum should not deviate from the vertical plane by more than 3 mm. The length of pendulum arm is measured from the rocking axis of the pendulum to the axis of weight.

The speed of pendulum at impact should be from 4 m/s to 7 m/s. The impact work of fracture of the specimen should be measured with an accuracy not worse than 0.98 J. The specimen should lie tightly on the supports and the temperature in the room should be within 20-25°C.

Fracture of a specimen requires a certain amount of mechanical energy, i.e. a definite mechanical work is performed. The resistance to fracture in impact loading is measured by the quantity of work spent for fracture of a specimen, related to the cross-sectional area of the specimen at the notch. The cross-sectional area of specimen at the notch

$$S = bh_1$$

where b is the width of the specimen, and h_1 is its height.

The design height to which the pendulum is lifted is found as

$$H_d = H - h$$

where H is the height from the upper position of the weight to the specimen axis, and h is the height of deviation of the weight on performing the work.

In tests, the weight is lifted by an angle α and deviates by inertia by an angle β. Noting these angles, we find for the design height:

$$H_d = H - h = l\,(\cos\beta - \cos\alpha)$$

The work A performed by the pendulum

$$A = GH_d = Gl\,(\cos\beta - \cos\alpha)$$

Dividing A by the area of the notched portion of specimen, we obtain:

$$\frac{A}{S} = a_n$$

where a_n is called the impact toughness of a material. Substituting for A and S, we get:

$$a_n = \frac{Gl(\cos\beta - \cos\alpha)}{bh_1}$$

Impact toughness is measured in joules per square metre.

7.4. Compression and Bending Tests

Compression tests are widely used in the design of buildings and building structures, foundations, frames, housings and other elements of machines, bearings, supporting structures, etc.

Compression tests are mostly carried out for many non-metallic materials, such as concrete, brickwork, stone, wood and some polymers, and for some metals, such as cast iron, high-strength steels, and titanium alloys.

The main criteria of resistance of materials in compression tests are the ultimate strength in compression and relative deformation (relative contraction).

Some metals, such as pure aluminium or lead, possess a very high ductility and do not break in compression tests, i.e. it is impossible to determine their ultimate strength in compression.

The behaviour of brittle materials in compression is essentially as follows.

If, for instance, a brick is placed into a testing press, it will break at a certain load and the press load then drops down sharply. Fragments of a fractured brick have the shape of small pyramids with side faces directed at certain angles to the centre of the specimen.

A cylinder of mild steel acquires at a definite compression load a barrel-like shape and, as the load is increased further gradually, is flattened. Thus, for mild steels and

some other ductile materials it is practically impossible to determine the ultimate strength in compression.

Specimens of cast iron for compression tests have a diameter $D = 6$ mm and height $H = 6$ mm or $D = 10$ mm and $H = 15$ mm. In all cases, the dimensions of cylindrical specimens for compression tests are chosen so as to satisfy the inequality:

$$1 < \frac{H}{D} < 3$$

Specimens of stone, concrete, glass and wood are made of a cubic shape. In specimens of composite materials and wood, the orientation of fibres in the structure should be considered. The end faces of specimens should be machined carefully.

In some cases, the elastic modulus and deformations of metals are determined in compression tests of elongated cylindrical specimens of a height $H = 8D$, with thickened end portions of a diameter $2D$. The end faces of these specimens are ground to make them perfectly parallel to each other.

Compression tests can be carried out in all-purpose reversible rupture testing machines, Gagarin's presses, and special testing machines. Large structures, such as columns, brick walls, concrete plates, etc. are tested in special hydraulic presses of a loading capacity of an order of a few thousands of kilonewtons.

Special care should be given in compression tests to the load transfer from the machine to specimen, so as to distribute the forces uniformly over the end face area and apply the load strictly in the centre of specimen. Guide sleeves and spherical supports are used for the purpose. Skewing of specimens is not allowed. The load is measured by dynamometers and the displacements, by dial indicators and deflection meters. Besides, the stress-strain (contraction) diagram is recorded.

High friction between the plates of a press and the end faces of a specimen can result in non-uniform distribution of stresses in cross-sectional planes. Friction is reduced by lubricating the surfaces and using pads of a soft material.

Compression tests are carried out for determining the ultimate strength in compression σ_c and the relative contraction ε of a specimen. The ultimate strength in compression is determined by the formula:

$$\sigma_c = \frac{F_{\max}}{S_0}$$

where $F_{\max}$ is the maximum load in the test, and S_0 is the initial cross-sectional area of the specimen.

The relative contraction of a specimen on compression

$$\varepsilon = \frac{h_0 - h}{h_0} 100\%$$

where h_0 is the initial height of the specimen, and h is its height after testing (Fig. 148*a*).

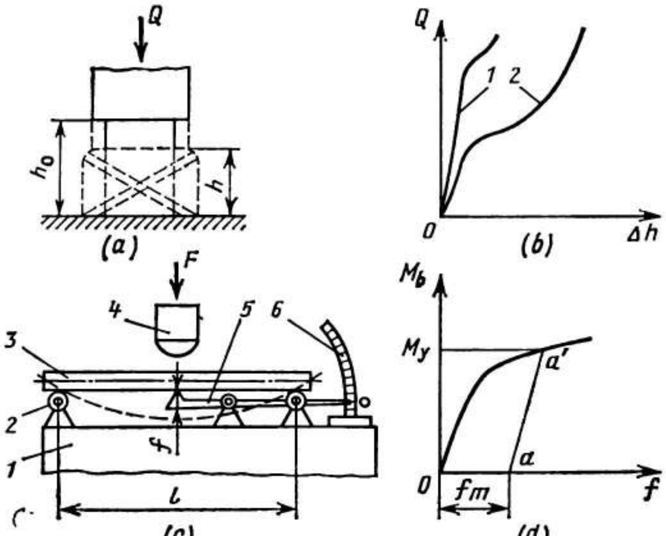


Fig. 148.
Compression and bending tests
(*a*) scheme of compression loading of a specimen; (*b*) diagram of state in compression; (*c*) scheme of bending loading; (*d*) diagram of state in bending

Values of ultimate strength in tension and compression for selected materials are given in Table 9.

In the design of machine elements, it is essential to consider the difference between σ_t and σ_c in order to

Table 9

Material	σ_t, MPa	σ_c, MPa
Cement	4	40
Polystyrene	15	55
Cast zinc alloys	35	300
Cast aluminium alloys	40	300
Cast iron	40	350
Wood (oak, along fibres)	100	27

avoid asymmetric loading of structures. For instance, the tensioned zone in a cast iron beam should be greater than the compressed zone; in a wood beam, on the contrary, it should be smaller than the compressed zone.

Figure 148*b* shows the compression test diagrams for cast iron (curve *1*) and annealed steel (curve *2*). As may be seen, cast iron exhibits a fracture point, whereas a steel specimen deforms gradually up to flattening.

If a beam is loaded by a lateral force F and the beam material obeys Hooke's law (Fig. 148*c*), the distribution of stresses in a section of the beam is described by a straight line having a certain zero point in which the material is neither tensioned nor compressed and the stresses are equal to zero.

This point lies on the neutral axis of the beam which passes through the centre of gravity of the cross section. Longitudinal tensile and compressive stresses in bending are directly proportional to the distance from the neutral axis of beam. Structures subjected to bending are designed so as to obtain the highest stiffness and minimize the weight. This is achieved by using cross-sectional shapes in which less material is located at the neutral axis and more at the edges of a cross section.

Specimens for bending tests have a diameter of 30 mm and length 340 mm or 680 mm, with the span between the supports respectively 300 mm or 600 mm. The free ends of specimens beyond the supports should be equal to 20 mm for 340-mm specimens and to 40 mm for those of a length of 680 mm.

Bending tests can be carried out in any type of testing machine. The load applied should be measured with an accuracy of 1%.

In bending tests (Fig. 148*c*), a specimen *3* is placed onto the supports *2* of the testing machine and a concentrated load *4* is applied in the mid of its span.

The deformation in bending appears as the deflection f of the beam, which is measured by a deflectometer *5* and indicated on scale *6*. The deflection can also be found in the load-deflection diagram taken during the tests (Fig. 148*d*). The diagram can be recorded automatically only under the action of a concentrated load. In cases of distributed load, the load-deflection diagram is constructed by experimental points. After the bending tests, the recorded diagram is checked for the presence of a yield step and the yield point is calculated.

The maximum bending moment is found by the formula:

$$M_{b\,\max} = \frac{Fl}{4}$$

where F is the concentrated force applied to the specimen, and l is the working length (span) of the specimen between the supports.

The maximum bending stress

$$\sigma = \frac{M_{b\,\max}}{W}$$

where W is the moment of resistance of the cross-sectional area of the specimen to bending.

The deflection caused by a concentrated load

$$f = \frac{(M_{b\,\max})l^2}{12EJ}$$

where J is the moment of inertia of the cross-sectional area of the specimen.

The yield strength in bending is found by the formula:

$$\sigma_{yb} = \frac{M_y}{W}$$

where M_y is the bending moment at the yield strength. It can be found easily if there is a yield step on the diagram. Otherwise, the bending moment at the yield point is determined as follows.

A deflection f_m is specified at which the deformations of the extreme fibres in bending are equal to 0.2%, i.e. as in tension. This deflection is laid off in the stress-strain diagram and a point a is found (Fig. 148*d*). Then a straight line parallel to the initial portion of the diagram is drawn from point a to the intersection with the curve, which gives a point a'. A horizontal line is then drawn through this point up to its intersection with the axis of ordinates, which gives the sought-for moment M_y.

The deflection f_m can be calculated by the formula:

$$f_m = \delta_r \frac{l^2}{6h}$$

where δ_r is the allowance for residual deformation relative to the extension of external layers (fibres), and h is the height of the specimen.

With M_y being known, it is now possible to find σ_{yb}.

7.5. Shear and Torsion Tests

Threaded, welded, riveted and cemented joints and some kinds of connecting parts, such as keys, inserts and shackles, are often subjected to shear loads. The pattern of deformation of a riveted joint under the action of a tensile load F is shown in Fig. 149*b*. A shear stress τ_{sh} then appears in the rivets.

Specimens for shear tests are made of a round or rectangular cross section. Figure 149*a* shows the scheme of testing of a rectangular specimen *2* by a load F applied to two knives *1*. Shear tests can be carried out under the same conditions and in the same types of testing machine as compression tests; it is only required to use special attachments.

A kind of such attachment for double shear tests of cylindrical pins is illustrated in Fig. 149*c*.

A specimen *4* is put into hardened-steel cylindrical inserts. The attachment also has a hardened inner sleeve *6*.

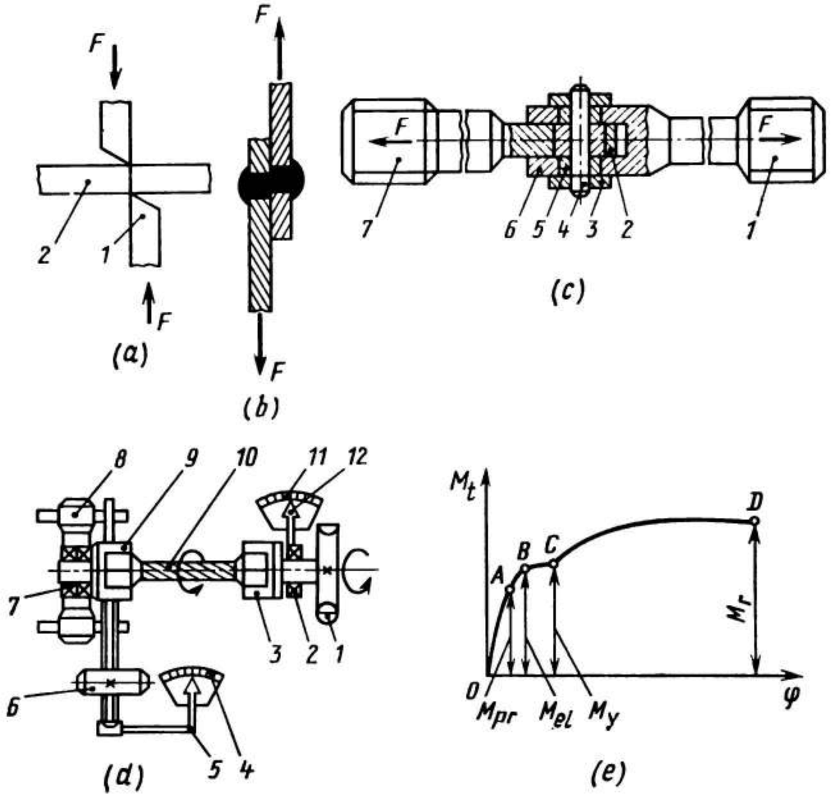


Fig. 149.
Torsion and bending test
(*a*) scheme of shear loading (*1* — knives; *2* — flat specimen); (*b*) scheme of loading of a riveted joint; (*c*) clamping mechanism for double-shear test (*1* — lower clamping head; *2* — internal sleeve disc; *3* — clamping ring; *4* — specimen; *5* — intermediate sleeve; *6* — internal sleeve; *7* — upper clamping head)

hardened intermediate sleeve *5*, and hardened clamping ring *3*. The inner sleeve is put into the disc of the upper threaded head *7* and the intermediate sleeve is connected to the shackle of the lower threaded head. The attachment is clamped in the clamps of a rupture testing machine and a load is applied to its heads.

The shear stress is calculated by the formula:

$$\tau_{sh} = \frac{F_{\max}}{S_0}$$

where S_0 is the initial cross-sectional area of the specimen, and F_{max} is the load at rupture.

With double shear, as in the case considered, the double cross-sectional area is taken:

$$\tau_{sh} = \frac{F_{max}}{2S_0}$$

A simple kind of loading is torsion which is effected by applying two equal but oppositely directed moments in finite sections. This kind of loading appears in joints of machine elements operating in rotation.

Torsion is a kind of deformation in which the longitudinal axis of a specimen is not curved and all points in a cross section of the specimen are turned through a certain angle relative to the longitudinal axis.

Torsional tests are carried out only with specimens of a round cross section, of a diameter $D = 10$ mm and the design length 50 mm or 100 mm. A control mark is made on the surface of a specimen parallel to its longitudinal axis.

The following rule is essential for making specimens for torsional tests: the higher the ultimate tensile strength σ_t of the material, the better should be the surface finish of specimens.

The shape of specimen heads depends on the design of machine clamps and may be square, cylindrical, or cylindrical with a flat. Torsional tests are mostly carried out in horizontal testing machines developing a torsional moment of a few hundred kilogram-metres.

Figure 149*d* shows the scheme of torsional test of a round specimen with square heads. A specimen *10* is fastened by its square heads in clamps *3* and *9*. The first clamp, *3*, rotates in a bearing *2* which is connected with pointer *12* of scale *11*. Rotation is transmitted to the clamp from a drive through a worm reducer *1*. The left-hand movable clamp *9* moves on guides *8* arranged parallel to the specimen length.

The torsional moment of the specimen is counterbalanced by a pendulum *6* which rotates together with the left-hand clamp on bearing *7*. The turning angle is

indicated by pointer *12* on scale *11*. The torsional moment is controlled by the indication of pointer *5* on scale *4*.

The results of the tests are used for plotting the torsional moment-turning angle diagram (Fig. 149*e*). The diagram for annealed steel which is shown in the figure, has the characteristic points A, B, C, and D which correspond respectively to the moment of proportionality, moment of elasticity, moment of yield, and moment of ultimate rupture strength: M_{pr}, M_{el}, M_y, and M_r. These moments are associated with the stresses τ_{pr}, τ_{el}, τ_y, and τ_r.

The results of torsional tests serve for determining the relative shift in torsion:

$$\gamma = \frac{\varphi_1 - \varphi_2}{2} \frac{d}{l} 100\%$$

where φ_1 and φ_2 are the turning angles at the ends of the nominal length of specimen, radians; d is the diameter of the design portion of specimen, mm; and l is the nominal length of specimen, mm.

The shear modulus in torsion is found as the ratio of the tangential stress in the elastic region to the relative shift:

$$G = \frac{Ml}{J_p (\varphi_1 - \varphi_2)}$$

where M is the torsional moment, and J_p is the polar moment of inertia of a cylindrical specimen:

$$J_p \approx \frac{\pi d^4}{32}$$

The proportionality limit in torsion

$$\tau_{pr} = \frac{M_{pr}}{W_r}$$

where W_r is the moment of resistance; for a cylindrical specimen

$$W_r = \frac{\pi d^3}{16}$$

There are certain methods for determining the yield point and ultimate strength in torsion whose description is however beyond the scope of this book.

The correlation between the modulus of elasticity and shear modulus is as follows:

$$G = \frac{E}{2(1+\mu)}$$

$$E = 2G(1+\mu)$$

7.6. Fatigue Tests

Some structures are subjected in operation to loads which vary repeatedly and act for a long time. Such loads are called cyclic. Structures and specimens operating under cyclic loads can break well before the ultimate srtength of the material is attained. The fracture of a material after a large number of cyclic loads applied to it is called fatigue and the ability of a material to withstand such stresses without fracture is called fatigue strength.

Fatigue fracture is initiated by the appearance of numerous micro-fissures in the microstructure of material. These fissures can originate at various defects of the material, such as voids, surface roughness, cracks caused by heat treatment, various inclusions, structural inhomogeneities, and sharp changes of shape.

These defects can serve as 'exciters' of plastic deformations in the material. In further loading, micro-fissures appear in the points of 'excitation', which then grow wider until fracture takes place.

After fracture, the surface of a metal specimen has a smooth 'lapped' zone and a zone where the crystalline structure of the metal is clearly seen. In the plane of breakage, fatigue fracture has typical concentric rings and strips.

Fatigue tests of structures and specimens are carried out by applying sign-varying loads and determining the fracture stress.

The pattern of loading can be represented by a periodic curve (Fig. 150). Fatigue tests are carried out by using the concept of cycle.

If a load or stress varies continuously and periodically about a certain mean value, a single period of its variation is called a cycle. The number of stress cycles per second is termed the frequency of stress variation.

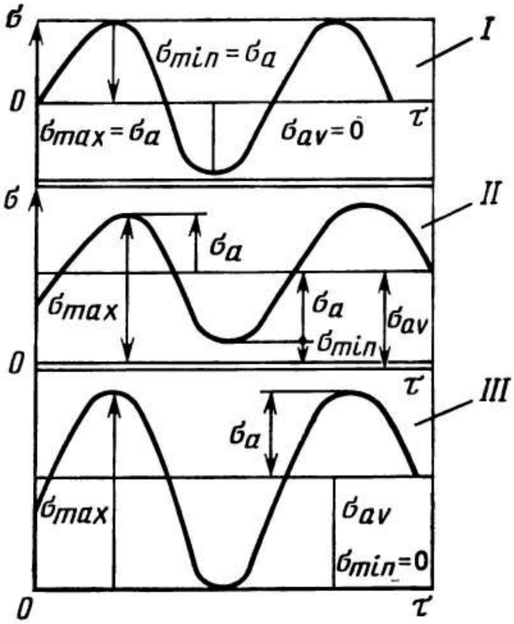


Fig. 150.
Types of cycles in fatigue tests

In fatigue tests, it is distinguished between three typical loading cycles (see Fig. 150): symmetrical (*I*), asymmetrical (*II*), and pulsating (*III*).

Depending on its type, a cycle can be characterized by the maximum normal stress σ_{max}, minimum normal stress σ_{min}, average stress σ_{av}, and amplitude stress σ_a and by corresponding tangential stresses τ_{max}, τ_{min}, τ_{av}, and τ_a.

Let us consider an asymmetrical cycle *II* (see Fig. 150) as the most general case of cyclic loading.

The stresses acting on the specimen vary in a cycle between two extreme values: the maximum stress σ_{max} and the minimum stress σ_{min}. Therefore, the average stress

can be found from them. For normal stresses, it is found by the formula:

$$\sigma_{av}=\frac{\sigma_{\max}+\sigma_{\min}}{2}$$

and for tangential stresses, by the formula:

$$\tau_{av}=\frac{\tau_{\max}+\tau_{\min}}{2}$$

As may be seen from the formulae, the average stress in a cycle is the algebraic half-sum of the maximum and the minimum stress in the cycle.

Let us now find in the graph the cycle amplitude and determine the amplitude stresses:

for normal stresses

$$\sigma_{a}=\frac{\sigma_{\max}-\sigma_{\min}}{2}$$

and for tangential stresses

$$\tau_{a}=\frac{\tau_{\max}-\tau_{\min}}{2}$$

Thus, the amplitude of a cycle, or the amplitude stress, is the algebraic half-difference between the maximum and the minimum stress of the cycle.

A cycle can also be characterized by what is called the coefficient of asymmetry r which is determined by the following formulae respectively for normal and tangential stresses:

$$r=\frac{\sigma_{\min}}{\sigma_{\max}}, \qquad r=\frac{\tau_{\min}}{\tau_{\max}}$$

i.e. the coefficient of asymmetry of a cycle is the algebraic ratio of the minimum stress to the maximum stress.

It should also be noted that two kinds of variable dynamic loads can act in fatigue tests. The first kind corresponds to symmetrical loading, i.e. the force applied varies symmetrically about the horizontal axis, say from $+F$ to a force of the same magnitude, but opposite sign, $-F$. In all other cases, i.e. in asymmetrical and pulsating loading, the load varies from $F_{\max}$ to $F_{\min}$, but these forces are not equal in magnitude to each other.

As may be seen from the curves in Fig. 150, an asymmetrical cycle (*II*), the most general case of cycle loading, is characterized by σ_{max}, σ_{min}, σ_a, and σ_{av}. In a pulsating cycle (*III*), which is a particular case of asymmetrical cycle, $\sigma_{min} = 0$. In a symmetrical cycle (*I*), $\sigma_{av} = 0$.

Any stress cycle can be obtained by superposition of a particular symmetrical cycle (*I*) onto a constant average stress.

The maximum and minimum normal stresses are:

$$\sigma_{max} = \sigma_{av} + \sigma_a$$

$$\sigma_{min} = \sigma_{av} - \sigma_a$$

and the maximum and minimum tangential stresses

$$\tau_{max} = \tau_{av} + \tau_a$$

$$\tau_{min} = \tau_{av} - \tau_a$$

The work of a material in cyclic loading is estimated in terms of the fatigue limit (or endurance limit) which is understood as the maximum stress that the material can withstand an unlimited number of cycles for a given cycle asymmetry r. For symmetrical cycles, the fatigue limit is designated σ_{-1}.

Fatigue tests are carried out in order to determine experimentally the fatigue limit, i.e. the stress that can be applied repeatedly N times for a given kind of cycle without fracture of the material.

The number of cycles that can be repeated without fracture is designated N and usually specified at the level of 10^6, 10^7 or 10^8.

Fatigue tests are usually carried out on a series of 5 to 10 specimens which may be of the cantilever or common type.

A cantilever specimen (Fig. 151*a*) has a single clamping head of a length l_h. Its working length is l, total length, l_0, and diameter D. The specimen end opposite to the clamping head has a threaded hole.

The diameter D of specimens may be different and should be close where possible to that of natural structures. This is associated with the fact that the fatigue

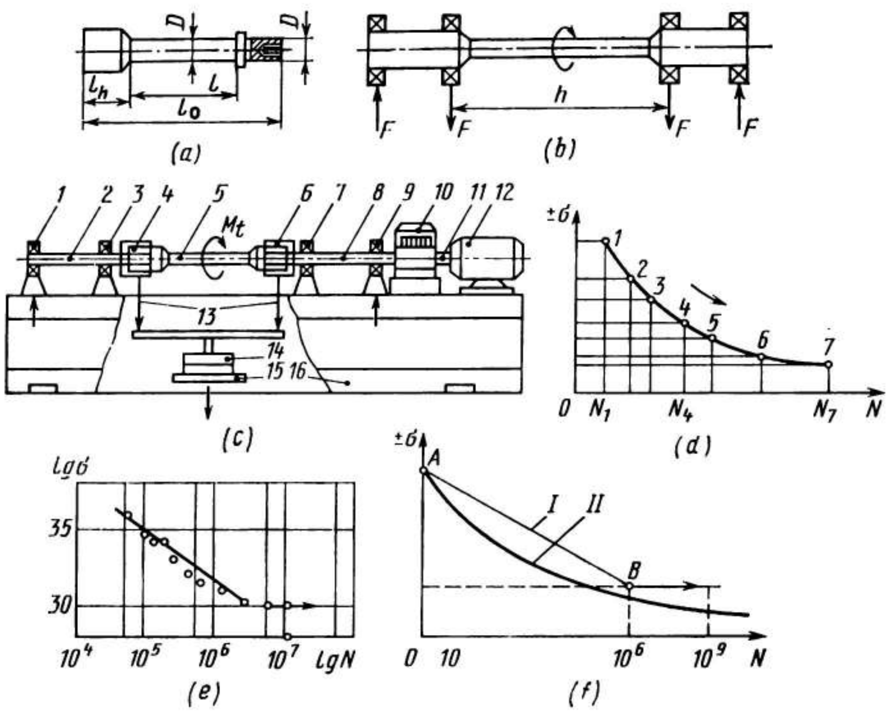


Fig. 151.
Fatigue tests

(*a*) cantilever specimen; (*b*) specimen with two heads; (*c*) scheme of fatigue-testing machine; (*d*) diagram of state

limit measured in experiments depends on the diameter of specimens.

A specimen for fatigue tests in pure bending is illustrated in Fig. 151*b*. It has two end portions (heads) for fastening in the clamps of a testing machine. The length-to-diameter ratio of specimens may range from 8 to 12.

Specimens are subjected to fatigue tests in special machines where tensile, compressive, bending and torsional cyclic forces can be developed. Fatigue tests under bending loads are most popular. A specimen is rotated in the machine from an electric motor and the cyclic load is developed by weights.

The scheme of fatigue tests in pure bending is shown in Fig. 151*c*. A specimen *5* is fastened in the clamps of

spindles *2*, *8* and rotated from electric motor *12* through flexible shaft *11*. The load is transmitted to the specimen by ties *13* through bearings *1*, *3*, *7*, and *9*. Tray *15* with weight *14* is attached to the ties. The number of shaft rotations is counted by counter *10*. The electric motor shaft rotates with a frequency from 1400 rpm to 3000 rpm. The testing machine is arranged on foundation *16*.

The results of the tests are used for plotting the diagram in coordinates: maximum stress σ—number of cycles N (Fig. 151*d*). In the upper portion of the curve, each stress σ has a corresponding number of cycles which a specimen can withstand to fracture. The lower portion of the curve approaches asymptotically a certain stress $\sigma = \sigma_{-1}$ which is exactly the fatigue limit of the tested material. The values of fatigue limit are considered to be established if the following inequality is satisfied:

$$|\sigma_{min} - \sigma_{max}| \leqslant 5\%$$

where σ_{min} is the lowest stress which has caused fracture, and σ_{max} is the highest stress that caused no fracture.

It is sometimes more expedient to represent the $\sigma - N$ relationship in semilogarithmic (logarithmic scale only for N) or logarithmic coordinates.

A logarithmic fatigue curve constructed by the results of tests of steel specimens is shown in Fig. 151*e*. As may be seen, the curve becomes horizontal at $\sigma = 30$ kgf/mm^2 (300 MPa) which is essentially the fatigue limit of the steel at $N = 10^7$. This type of curve is called Wöhler's curve. The fatigue limit is determined on Wöhler's curve as the ordinate at which the curve becomes parallel to the axis of abscissae.

The experimental fatigue curves for steel and aluminium specimens are shown in Fig. 151*f* (respectively curves *I* and *II*). A steel element will operate reliably indefinite time if it has not failed up to point B in curve *I*. Curve *II*, which has been obtained for an aluminium alloy, has no pronounced fatigue limit; the latter is determined conditionally by assuming a certain ordinate-to-abscissa ratio.

7.7. Creep Tests

It has been observed in natural structures that the stresses and strains can vary in them if a constant tensile or compressive load is applied for an appreciably long time.

A more detailed analysis of this phenomenon has revealed essentially two effects. The first is associated with time variations of strain and has been called creep, or aftereffect. The second relates to time variations of stresses and is called stress relaxation.

Aftereffect may be elastic or plastic. With elastic aftereffect, the deformation accumulated in time decreases after load releasing and can even disappear gradually. With plastic aftereffect, the deformations are irreversible and decrease only insignificantly after load releasing.

The process of elastic aftereffect under the action of a stress that is constant in time and is substantially lower than the proportionality limit of the material at the temperature of tests is illustrated graphically in Fig. 152*a*. The deformation *O-A* which appears on initial loading and is equal to σ/E, increases in time up to the deformation corresponding to point B. During load releasing, the deformation decreases from point B to C again by σ/E and then decreases from point C to D, i.e. to zero. This phenomenon has been called the reverse aftereffect, or reverse creep.

In the case of plastic aftereffect (Fig. 152*b*), the process of deformation occurs like in elastic creep up to point B. Beginning with point C, however, the deformation does not diminish to zero, as in the former case, but varies gradually to point D which is characterized by a definite residual deformation.

In the course of creep, plastic deformation increases to a magnitude at which fracture of the material takes place.

Figure 152*d* shows a curve of relaxation, i.e. of a process in which the stresses vary in time without any change in the deformations. In the course of relaxation, the plastic deformation does not exceed the deformation formed on loading of the material.

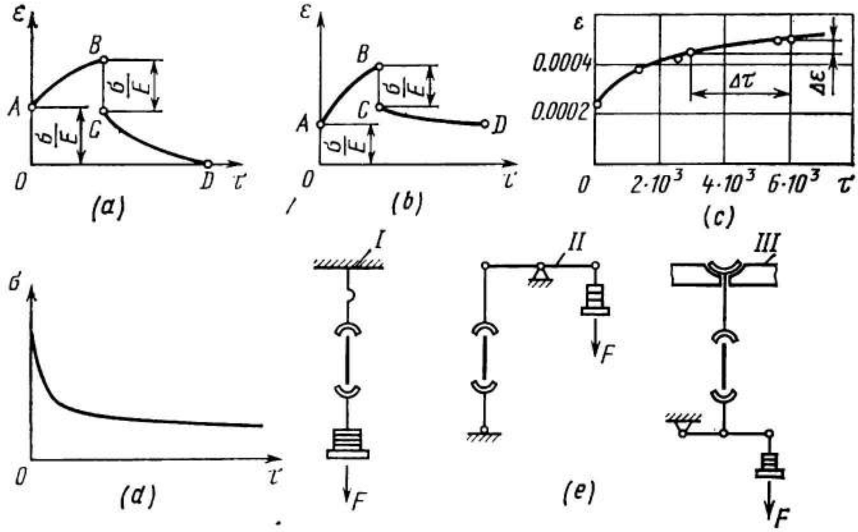


Fig. 152.
Creep tests

(*a*) diagram of state with elastic aftereffect; (*b*) diagram of state with plastic aftereffect; (*c*) diagram for determining the strain rate; (*d*) relaxation curve; (*e*) loading diagram

In tensile creep tests, the initial load is diminished in time so that the length of the specimen remains constant and corresponding to that formed by the initial load.

The probability of creep should be taken into account for elements which have to operate for a long time at elevated temperatures, such as gas turbine blades, elements of steam boilers, etc. The creep phenomenon is especially noticeable in the metals having a low melting point, such as lead or aluminium, and in polymers.

Specimens for tensile creep tests usually have a cylindrical shape with thickened end portions (heads) for clamping in the testing machine. In creep tests, it is essential to ensure a constant load, specified temperature conditions, and the possibility for measuring small displacements.

Schemes of creep tests are illustrated in Fig. 152*e*: *I*—scheme with direct loading by a static weight; *II*—with loading through a lever; and *III*—with loading through a system of levers.

In creep tests at elevated temperatures, the temperature field is created by means of an electric heater (furnace) which should ensure uniform heating along the entire length of a specimen. The temperature in the furnace is maintained constant within 1-2 degrees C by means of a temperature controller. Temperature is measured by means of nickel chromium thermocouples at three points along the specimen length.

Deformations of specimens during the tests are measured by thermostable strain gauges bonded to their surface.

Creep tests are carried out for determining the creep limit, i.e. the stress at which the rate of creep at a given temperature during a specified time interval will not exceed the specified limit.

The results of tests are used for constructing the strain-time (ε-τ) curves for a number of tested specimens, for instance, made of high-temperature steel (see Fig. 152*c*). The relative rate of uniform creep can be found by the formula:

$$v = \frac{\varepsilon}{\Delta\tau} = \frac{\Delta l}{l\Delta\tau}$$

where l is the design length of a specimen; Δl is the length increment; and $\Delta\tau$ is the time increment.

A graph is then constructed, which correlates the calculated creep rates v and the stresses σ causing these rates (see Fig. 152*d*). The stress found on this curve for the specified creep rate is the creep limit under the specified conditions.

The relationship between the stress σ and strain ε at a time moment τ can be written as:

$$\varepsilon_\tau = \frac{\sigma_\tau}{E} + \Delta\varepsilon$$

where ε_τ is the strain of the body at time moment τ; σ_τ is the stress in the body at that moment; $\Delta\varepsilon$ is the deformation accumulated in the body up to the time moment τ; and τ is the time of observation.

Creep tests of specimens can also be carried out up to fracture. In that case, they determine what is called

the long-term strength σ_{lt}. The long-term strength is defined as the stress at which a specimen will fail in a specified time. It is determined as the ratio of the fracture load to the cross-sectional area of a specimen:

$$\sigma_{lt} = \frac{F^{\tau}_{\max}}{S_0}$$

The time of loading of a specimen is chosen equal to the time of real service life, i.e. the time of operation of the material under real conditions.

As has been demonstrated in experiments, creep can be influenced by stress concentrators. They can decrease the long-term strength.

7.8. Tests at High and Low Temperatures

Various structures, machines and machine elements are usually designed for operation in a particular temperature range. For instance, transport vehicles are designed for normal operation at temperatures from +40°C to —50°C, gas turbine blades operate at temperatures of a few hundred degrees, elements of cryogenic plants, at temperatures as low as —200°C, etc.

A change of temperature can change the mechanical properties of a material and the behaviour of the material in creep and fatigue tests. Changes in the temperature field can result in changes of the shape and dimensions of elements.

With an increase of temperature, the ultimate strength, proportionality limit, creep limit, and elastic modulus of a material decrease, whereas the strain increases. The plasticity of the material increases, resulting in a higher creep, and the surface of elements is subjected to more intensive oxidation.

At very low temperatures, the strength of metals increases, but their ductility diminishes. The relative elongation and relative contraction decrease substantially and can cause embrittlement. At lower temperatures, a material becomes more sensitive to stress concentrators.

The mechanical properties of materials at elevated and reduced temperatures are mainly determined by tensile tests of specimens.

Tensile tests at elevated temperatures are carried out in order to determine the mechanical characteristics of strength at specified temperatures. Specimens are similar to the standard specimens for tensile tests and are tested in standard testing machines, but they are heated during the test by an electric muffle furnace.

The quality of tests depends on the heating conditions for specimens and on the accuracy with which constant heating conditions are maintained. Constant heating conditions are formed by means of a tubular electric muffle furnace (Fig. 153*a*). A specimen *4* is threaded with its end to an upper tie *2* and lower tie *9* and is held in the muffle furnace *5* by means of covering plugs *7* and *3* which have heat-insulating ceramic inserts.

The length of uniform heating in a muffle made of ceramic or high-temperature steel should be 2-4 times the design length of a specimen and the latter's diameter is chosen according to the inequality $D > 3d$, where D is the diameter of the muffle, and d is the diameter of the specimen.

The muffle *6* together with specimen *4* is fastened by means of ties *2* and *9* in clamps *1* and *8* of the testing machine. The temperature conditions in the furnace are controlled by a plate-type temperature controller *10*. The temperature of the specimen is controlled by means of nickel-nickel-chromium or chromel-alumel thermocouples with terminals *11*.

Deformations of the specimen are measured by strain gauges. The compressive or tensile stresses in uniform heating or cooling of a material can be calculated by the formula:

$$\sigma = E\alpha\,(t_0 - t)$$

where α is the coefficient of linear expansion, and $t_0 - t = -\Delta t$ is the change of temperature relative to the initial temperature t_0. The formula for the stress will then be:

$$\sigma = -\alpha E\,\Delta t$$

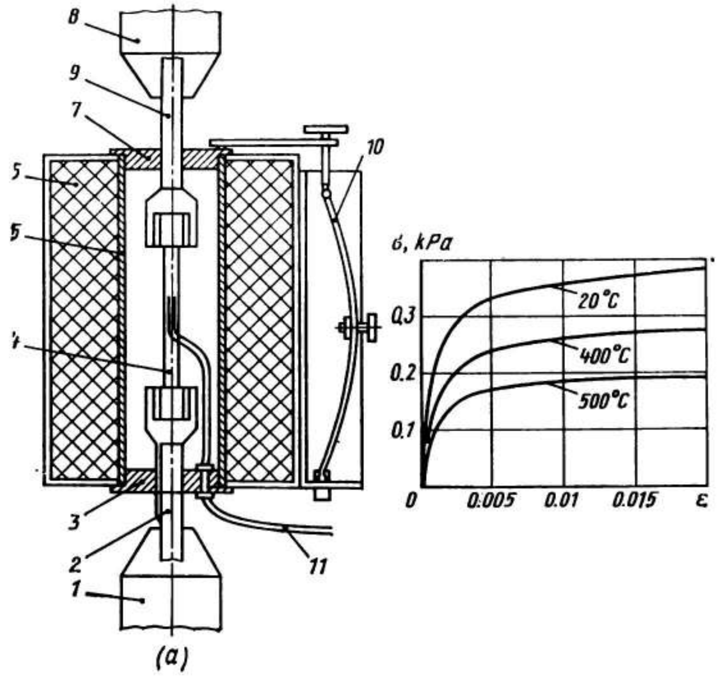


Fig. 153.
Tests at elevated temperatures

(*a*) muffle furnace with a specimen (*1* — lower grip; *2* — lower tie; *3*, *7* — cover plugs; *4* — specimen; *5* — tubular muffle furnace; *6* — electric furnace; *8* — upper grip; *9* — upper tie; *10* — plate-type temperature controller; *11* — thermocouple terminal); (*b*) diagram of state in heating of carbon steel

The results of tensile tests at elevated or reduced temperatures are represented graphically as a diagram with stresses laid off as ordinates and temperature variations, as abscissae.

The stress-strain diagrams for medium-carbon steel at various temperatures are shown in Fig. 153*b*. As may be seen from the diagram, at heating to 500°C, the stress causing a particular amount of deformation decreases almost by 50%. Relative values of the strength of selected materials in tensile tests at various temperatures are given in Table 10. As follows from the table, heating to 500°C decreases the ultimate strength: of steel by 43%,

Table 10

Material	Relative ultimate strength[1] at temperature, °C					
	20	100	200	300	400	500
Cast steel	1.00	1.09	1.26	1.21	0.97	0.57
Cast iron	1.00	1.00	1.00	0.99	0.92	0.76
Copper sheet	1.00	0.95	0.85	0.73	0.65	0.50
Bronze	1.00	1.01	0.94	0.57	0.26	0.18

[1] Ultimate strength at 20°C is taken as unity

of cast iron by 24%, of copper by 50%, and of bronze by 82%.

On reduction of temperature to —200°C, the ultimate strength and creep limit increase by 20-30% on the average.

7.9. Stability Tests of Bars

Bars of a large length and small cross-sectional area can lose stability when loaded by a longitudinal compressive force.

The essence of this phenomenon consists in that a bar changes sharply its initial shape when the axial compressive load applied to it attains a certain limit. In that case, the stresses appearing in the bar may be much lower than the ultimate strength and even lower than the elastic limit.

A laboratory arrangement for stability tests is shown in Fig. 154*a*. Two columns *10* carrying a fixed cross-piece *6* and movable cross-piece *4* are mounted on a steel plate *1*. The plate and movable cross-piece carry two supports, *2* and *9*, between which a specimen is mounted. The length of the specimen exceeds many times its cross-sectional dimensions.

The compressive load F is applied to the specimen by a handwheel *7* through a screw-nut pair, dynamometer *5*,

and movable cross-piece *4*. The force measured by the dynamometer is indicated on scale *8*.

The load is applied in steps. The deflection f_{ad} of the specimen is measured by a dial indicator *12* which is fastened by bracket *3* on one of the columns. The results

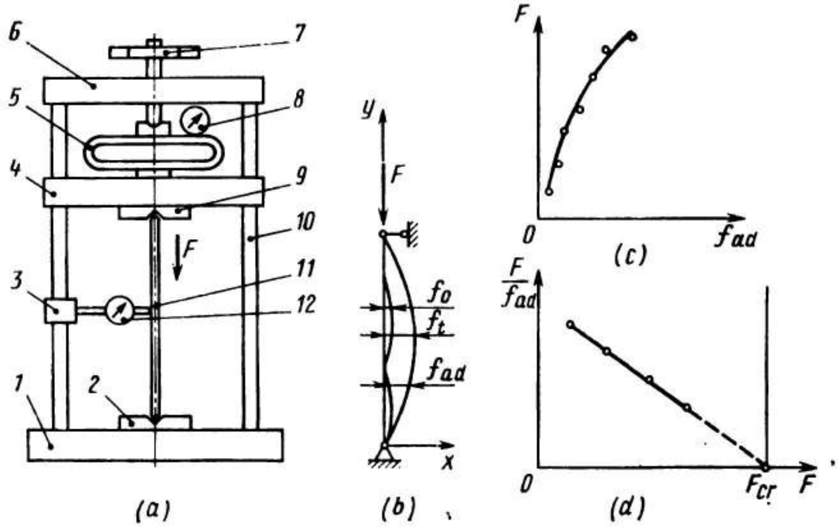


Fig. 154.
Stability tests

(*a*) scheme of apparatus (*1* — base plate; *2* — lower support; *3* — bracket; *4* — movable cross-piece; *5* — dynamometer; *6* — upper cross-piece; *7* — handwheel with screw and nut; *8* — indicator; *9* — upper support; *10* — stand; *11* — specimen; *12* — indicator); (*b*) scheme of bar loading; (*c*) load-displacement curve; (*d*) determination of the critical load

of the test are obtained as a number of values of compressive load F and corresponding additional deflection f_{ad} in the centre of the bar length.

The bar is loaded by a compressive force F whose line of action passes through the ends of the bar (Fig. 154*b*). The bar has an initial deflection f_0 which is different along its length. At the bar ends, the initial deflection is equal to zero.

On application of a load, the bar acquires an additional deflection f_{ad} and the total deflection is the sum of the

initial and additional deflections:

$$f_t = f_0 + f_{ad}$$

The aim of stability tests of bars is to determine the critical load in terms of the additional deflection of a bar. It should be noted that a bar with hinged fastening of ends will loose stability in one half-wave. For that reason, the deflection should be measured where it is at the maximum, i.e. in the mid of the bar length.

In order to determine the critical load, it is required first to construct a diagram relating the additional deflection and load (Fig. 154*c*).

This diagram is then re-constructed as follows (Fig. 154*d*). Dividing the load by the additional deflection, we find a number of values of F/f_{ad} which are laid off as ordinates. The loads are laid off along the axis of abscissae. The points obtained are connected by a straight line which is continued up to its intersection with the horizontal axis. The load thus obtained is exactly the critical load F_{cr}.

The initial deflection f_0 can be found as the tangent of the angle of incline between the straight line and the axis of abscissae:

$$f_0 = \frac{1}{\tan \alpha}$$

Thus, in the stability tests for an imperfect bar, we have obtained the critical load F_{cr} for an ideal bar.

The relationship between the total deflection and load is as follows:

$$f_t = \frac{f_0}{F/F_{cr}}$$

where F is the load at a stage, and F_{cr} is the critical load of an ideal bar.

This equation can be re-written for the additional deflection

$$f_{ad} = \frac{f_0}{\frac{F_{cr}}{F} - 1}$$

or

$$f_{ad}F_{cr} - f_{ad}F = f_0F$$

Dividing both parts by F, we get:

$$\frac{f_{ad}}{F} F_{cr} = f_{ad} + f_0$$

where $f_{ad} + f_0$ is the total deflection f_t.

As follows from this equation, f_{ad}/F and f_{ad} are in a linear relationship where F_{cr} is a proportionality factor. This equation can be represented in the following form:

$$\frac{F}{f_{ad}} f_0 = F_{cr} - F$$

which makes it possible to construct a diagram in the coordinates F/f_{ad} versus F (see Fig. 154*d*) and to find from it immediately the critical load F_{cr}.

Any real bar usually has certain deviations from the ideal shape which may be called initial imperfections. In order to find an initial imperfection f_0, the formula given above can be re-written relative to f_0:

$$\frac{f_{ad}}{F} F_{cr} = f_{ad} + f_0$$

after which a graph of f_{ad}/F depending on f_{ad} can be constructed. Thus, we can determine graphically the initial imperfection f_0. The critical load determined in this way is exactly Euler's critical load determined in stability tests of a bar.

Review Questions

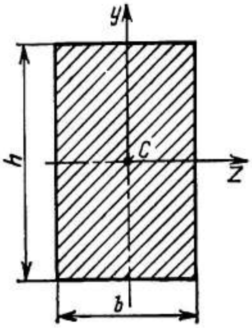
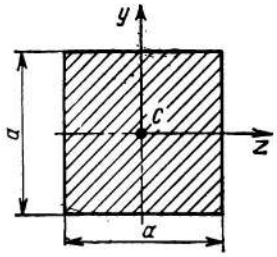
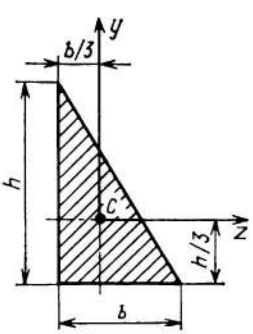
1. What kinds of loading do you know?
2. What is meant by static tests?
3. What is meant by dynamic tests?
4. Name special kinds of tests.
5. What are the ultimate strength, the elastic limit, the creep limit, the proportionality limit?
6. What is relative deformation?
7. What is relative contraction?
8. How are specimens of sheet materials cut out?
9. How is the design length of a specimen determined?

10. What is strain measurement?
11. What is a strain gauge?
12. Explain the circuit of a resistance strain gauge.
13. Explain the design of a mechanical strain gauge.
14. What types of testing machines do you know?
15. How are specimens fastened in a testing machine?
16. What is the hardness of a material?
17. Explain the Brinell hardness test.
18. Explain the Rockwell hardness test.
19. Explain the Vickers hardness test.
20. How is the modulus of elasticity determined?
21. What is Poisson's ratio?
22. How are dynamic stresses determined?
23. What kinds of specimens are manufactured for impact tests?
24. How is the impact toughness calculated?
25. How is the ultimate strength in compression determined?
26. What is the behaviour of materials in compression?
27. What kinds of strain can appear in compression?
28. What kinds of strain can appear in bending?
29. How are bending stresses determined?
30. How can the deflection of a beam be measured?
31. What kinds of stress appear in shear?
32. What stresses are determined in torsion?
33. How is the shear modulus determined?
34. What is the fatigue strength of a material?
35. What stresses are determined in fatigue tests?
36. How is the fatigue strength influenced by stress concentrators?
37. What is creep?
38. What is relaxation?
39. What is long-term strength?
40. What is the effect of temperature on creep?
41. How are the strength characteristics of a material changed on heating and cooling?
42. What is embrittlement?
43. How is the plasticity of a material influenced by temperature?
44. What is the initial deflection?
45. What is the additional deflection?

46. What is the total deflection?
47. Explain the loading scheme in stability tests.
48. How is the critical load in stability tests determined?

APPENDIX 1

Moments of Inertia of Simple Geometrical Figures

Section	I_z	I_y
	$\frac{bh^3}{12}$	$\frac{hb^3}{12}$
	$\frac{a^4}{12}$	$\frac{a^4}{12}$
	$\frac{bh^3}{36}$	$\frac{hb^3}{36}$

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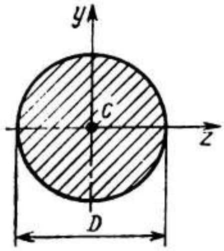
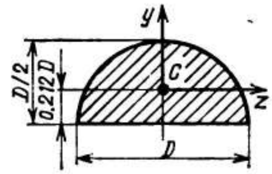
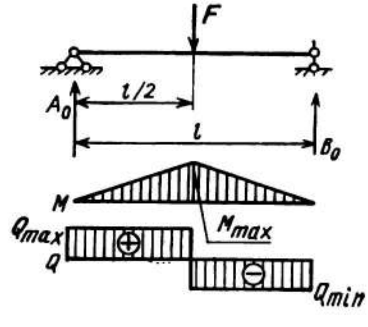
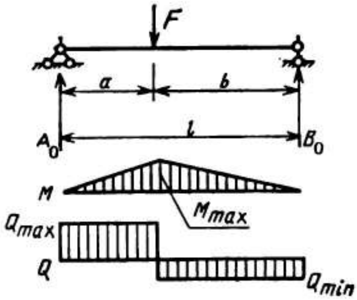
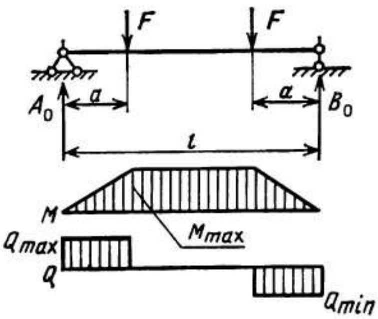
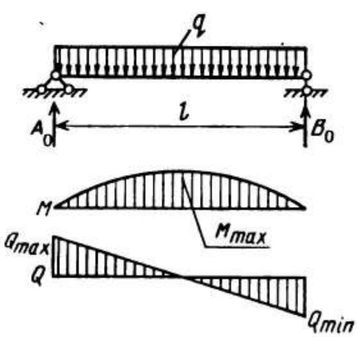
Section	I_z	I_y
	$\frac{\pi D^4}{64} \cong 0.05D^4$	$\frac{\pi D^4}{64} \approx 0.05D^4$
	$0.00686D^4$	$\frac{\pi D^4}{128} \approx 0.025D^4$

APPENDIX II

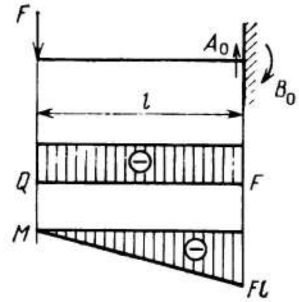
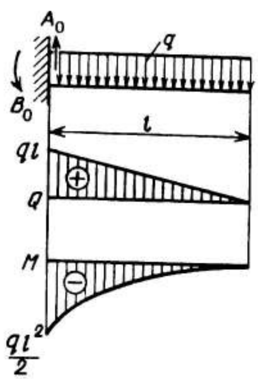
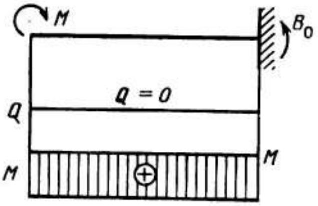
Support Reactions A_0 and B_0, and Diagrams of Q and M for Simple Beams

Design diagram, Q and M diagrams	A_0	B_0	$[Q_{max}]$	$[M_{max}]$
	$\frac{F}{2}$	$\frac{F}{2}$	$\frac{F}{2}$	$\frac{Fl}{4}$

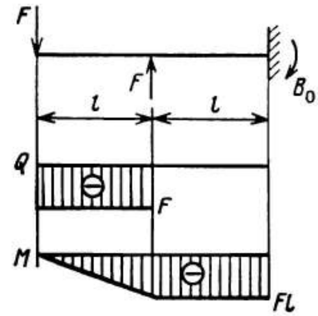
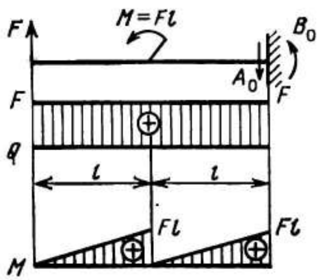
Continued

Design diagram, Q and M diagrams	A_0	B_0	$[Q_{max}]$	$[M_{max}]$
	$\frac{Fb}{l}$	$\frac{Fa}{l}$	$\frac{Fb}{l}$	$\frac{Fab}{l}$
	F	F	F	Fa
	$\frac{ql}{2}$	$\frac{ql}{2}$	$\frac{ql}{2}$	$\frac{ql^2}{8}$

Continued

Design diagram, Q and M diagrams	A_0	B_0	$[Q_{max}]$	$[M_{max}]$
	F	Fl	F	Fl
	ql	$\frac{ql^2}{2}$	ql	$\frac{ql^2}{2}$
	0	M	0	M

Continued

Design diagram, Q and M diagrams	A_0	B_0	$[Q_{max}]$	$[M_{max}]$
	0	Fl	0	Fl
	F	Fl	F	Fl

INDEX

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